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# THE NATURE OF MICROWAVE PARAMETRIC AMPLIFICATION



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UNITED STATES NAVAL ORDNANCE LABORATORY, WHITE OAK, MARYLAND

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# THE NATURE OF MICROWAVE PARAMETRIC AMPLIFICATION

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ABSTRACT: The principles of parametric amplification are discussed. Included are the conservation theorems that place restraints on powers at various frequencies, feedbacks operating internally, the physical mechanisms of pumping resonant circuits, and the physical mechanisms of pumping and coupling in propagating systems. The treatment is theoretical in nature: Known laws and relations are surveyed and the theory is extended to include the nature of the feedback, pumping, and coupling mechanisms. Feedback is found to be positive for the lower-sideband three frequency parametric amplifier, but also positive for the upper-sideband case where limited gain results from the feedback loop gain being less than unity. An entire spectrum of permitted pumping frequencies below the system resonant frequency, as well as above, has been found. Although the results are applicable to all parametric amplifiers, illustrations are taken from amplifiers in the microwave frequency range.

U. S. NAVAL ORDNANCE LABORATORY
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Under BuWeps Task No. RMWC-51-025/212-1/F009-04-001, entitled "Fire Control; Fire Control Techniques; Solid State Microwave Amplifiers; New Techniques," elementary components of solid state microwave amplifiers have been built and tested and the fundamental nature of microwave amplification has been studied.

This report covers fundamental principles of parametric amplification derived from the study of the nature of microwave amplification.

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Commander

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By direction

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# INTRODUCTION

We begin with a few comments about amplification in general. Specifically what is the gain mechanism of an amplifier or what is the essential requirement for obtaining an amplifier? An amplifier is "a device for increasing the power associated with a phenomenon without appreciably altering its quality, through control by the amplifier input of a larger amount of power supplied by a local source to the amplifier output."[1] Let us rephrase these ideas into a form more suitable for seeing the fundamental nature of the "increase" in power one expects from the amplification process. This will then be our working definition of an amplifier.

## Working Definition of an Amplifier

An amplifier is a device wherein an "agent" is driven by a local source of power, has motion imparted to it and hence gains energy of motion. This "agent" then by geometry or circuitry has its movement affected, and energy released in an external circuit by the expenditure of less energy in the input or controlling element. This is essentially a classical description of an amplifier. It will be most strained when used to describe the quantum mechanical amplifiers where energy of motion is imparted to the spin of a bound electron in a paramagnetic ion and the geometry or circuitry is relatively hidden emission process.

In a vacuum tube amplifier the "agent" is, of course, a free electron. The motion imparted to it is translational motion. Its kinetic energy is derived from the local source of power, i.e., the voltage across the anode to cathode spacing and the current in the anode circuit. The controlling element is the control grid which by geometry is capable of changing the kinetic energy of the electrons by a greater amount than the energy it consumes itself in the process.

## Local Power

While the local sources of power fall into two classes, direct power and alternating power, the agents employed are many and varied. The proper choice of power source depends on the agent employed. Let us look at this relationship for the two types of power sources. First, if the movement of this agent is free and unidirectional, the source of power can be direct. Examples are the free electron in the vacuum tube, the translational motion of electrons or holes in the semiconductor transistor amplifier and Esaki diode amplifier.

Second, if the movement of this agent is bound or constrained at some limit the source of power must alternate. An example is the use of domain walls as the agent in a magnetic amplifier where the wall motion is constrained by the dimensions of the material or by complete alignment with the field. Another example is the macroscopic precessing magnetization vector $\vec{M}$ (the vector sum of the atomic magnetic dipole moments per unit

volume) used as the agent in the microwave magnetic amplifier. Here the motion is centrally bound. Other examples are the charge on the capacitance of a P-N junction (the Varactor parametric amplifier), or the quantized spin orientations of the Maser. All of these agents, since they are bound or constrained, require alternating power to excite them. Chart I shows a classification of amplifiers as to direct local power and alternating local power. The nature of the amplifying "agent" and the approximate operating frequency range is indicated in each case. In this study we will confine ourselves to the alternating local power type of amplifier and call these collectively parametric amplifiers. This term is not always applied to such diverse devices such as the magnetic amplifier and the Maser. However, it will be obvious from the following treatment that we are dealing with basically common properties. Since we will discuss the alternating local power case of amplification in the remainder of this report a brief description here of the direct local power amplifiers is not out of place for the sake of completeness.

Direct Local Power Amplifiers

Amplifiers in this class include the vacuum-tube, transistor, Esaki diode, and Technetron. The vacuum tube amplifiers can be further classified into the triode (plus other multigrid tubes of the same general nature) and the velocity-variation tubes (the double resonator klystron, the reflex klystron, and the traveling wave tube). As mentioned previously, the agent by which energy is transferred from the direct power source to the signal frequency is the free electron in the vacuum tubes, or the free electron or hole in the semiconducting transistor, Esaki diode or Technetron. "A net transfer of energy is achieved because some means of electron selection and rejection, sorting or bunching is operative. The necessary criterion is either that more electrons interact favorably with the rf (signal) fields than interact unfavorably, or that electrons which interact favorably do so over a longer time interval than those which interact unfavorably."[2]

With the triode the free electron current is amplitude modulated by the signal field. The phase is arranged so that more electrons are in the grid-anode region when the signal field retards and extracts energy from them than where it accelerates them and gives energy to them. The same net result of the electrons giving more energy to the signal field than they take from it is achieved in the velocity variation tubes by velocity modulation rather than amplitude modulation.

In the transistor, Esaki diode or Technetron, excited by direct local power the conductance of the material is controlled by the signal either by direct injection of charge as in the transistor or by the influence of the E field on the semiconductor as in the Technetron. In the Esaki diode, a region of negative conductance is utilized. In the case of direct power there is only one frequency present, the signal frequency. This assumes the signal to be a sinusoid of one frequency which extends in time from minus infinity to plus infinity. Since more than one sinusoid is required for signaling, it is obvious that some modulation must be applied to the signal

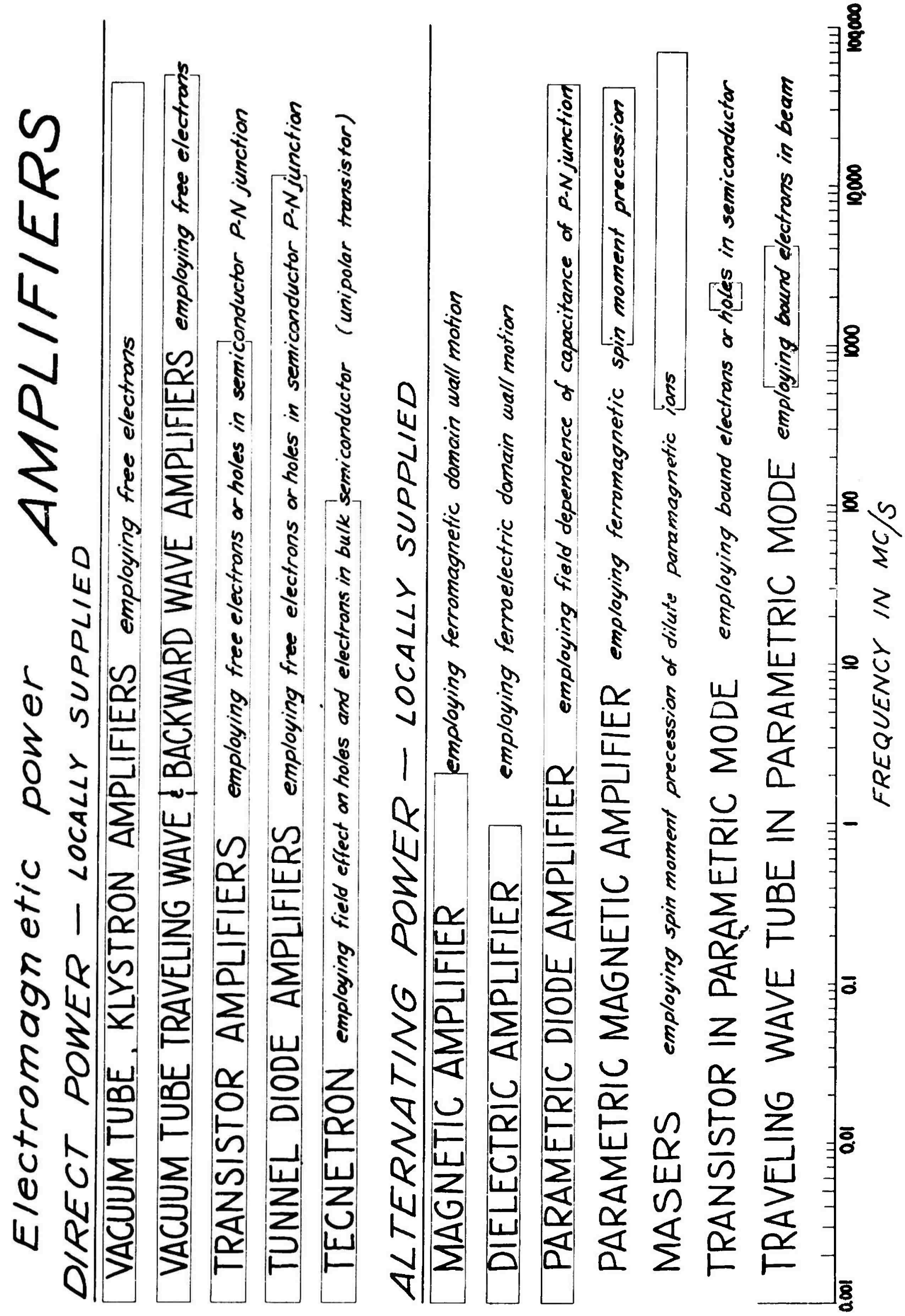



CHART 1 Classification of Amplifiers and Their Frequencies of Operation

frequency. For our purposes, however, we shall discuss the amplification of this single sinusoid. Assuming that the output-input relation is linear, then no new frequencies are introduced in the amplifying process and the output spectrum is identical with the input spectrum. We have high fidelity. If the output-input characteristic is non-linear, then harmonics of the signal frequency are created and the output spectrum is not identical with the input spectrum.

## Alternating Local Power

In the case of alternating power there are two frequencies present, the signal frequency and the frequency of alternation of the local power supply, sometimes called the local oscillator, beat frequency oscillator, carrier, or pump. Since two frequencies are present, modulation cross product terms are normally unavoidable. If the modulation cross product terms are absent in the a-c case then there has been no affected movement of any "agent." Many modulator devices, however, having cross product terms are still devoid of amplification. Consequently, the modulation process yielding cross product terms is a necessary but not sufficient condition for amplification in the a-c power case.

The conclusion is reached then that there is a distinction to be made between the mechanism which produces modulation cross-products and the mechanism which produces gain. One may have gain without modulation products in the case of d-c power, or one may have devices producing modulation products in the a-c case both with and without gain. This conclusion, which seems obvious is of considerable importance since the frequency-power relations that will be discussed in the a-c case will tend to obscure the fact of whether or not gain is present. The gain mechanism again, as in our working definition of an amplifier, is the process whereby the agent in the amplifier has its movement affected. This mechanism has some efficiency of converting local power to signal power, which we will call conversion efficiency. There are, however, also losses in the device. If losses exceed the conversion efficiency then there is no gain. If losses are sufficiently low and conversion efficiency high then gain is obtained.

## Modulation Theory

Since modulation cross product terms are seen to be a necessary but not a sufficient condition for amplification utilizing alternating local power, a brief review of modulation theory will be given. Modulation is the process of producing a wave, some characteristic of which varies as a function of the instantaneous value of another wave, called the modulating wave. In the broad sense, a wave is a periodic variation of an electric quantity of any frequency including zero, thus including as modulators all two frequency devices even where one "frequency" is zero. This case, however, produces cross-product terms equal to zero. Thus we will not treat this case and the distinction is made in this discussion between devices producing or not producing cross product terms.

The sinusoidal wave to be modulated is denoted by subscript c (for carrier) and the modulating wave is denoted by m. In addition, we assume that the amplitude of an rf modulated wave is directly proportional to the instantaneous value of the modulating wave. Then

$A_m \cos(\omega_m t + \beta_m)$ - - - - modulating wave

$A = A_c + k\left[A_m \cos(\omega_m t + \beta_m)\right]$ where $A_c$ = amplitude of wave before modulation and k is a constant

$i = A \cos(\omega_c t + \beta_c)$ - - - - wave to be modulated

$$i = \left[A_c + k\left[A_m \cos(\omega_m t + \beta_m)\right]\right] \cos(\omega_c t + \beta_c)$$

$$i = A_c\left[1 + m \cos(\omega_m t + \beta_m)\right] \cos(\omega_c t + \beta_c)$$

where $m = \frac{kA_m}{A_c}$ = modulation

$m \times 100$ = o/o factor modulation

This can be represented by a system of rotating vectors as in Fig. 1.

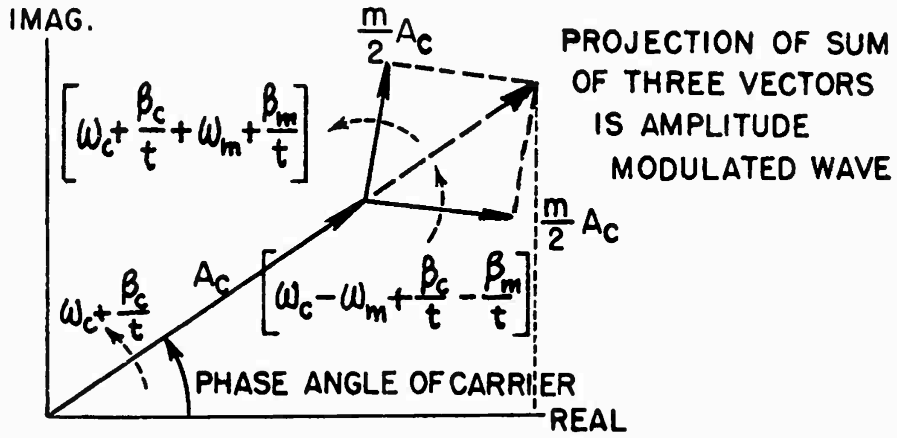



FIG. 1 VECTOR DIAGRAM OF AMPLITUDE MODULATION

$$i = A_c \cos(\omega_c t + \beta_c) + A_c m \cos(\omega_m t + \beta_m) \cos(\omega_c t + \beta_c)$$

since $\cos\alpha \cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$

$$i = A_c\Big[\cos(\omega_c t + \beta_c) + \frac{m}{2}\cos(\omega_m t + \omega_c t + \beta_m + \beta_c) \quad \text{Eq. 1}$$

$$\pm \frac{m}{2}\cos(\omega_c t - \omega_m t + \beta_c - \beta_m)\Big]$$

Where the first term in the brackets is the carrier, the second term is the

upper sideband and the third term is the lower sideband. The relative amplitudes of carrier, upper sideband and lower sideband are 1, $m/2$, $m/2$, respectively. The relative powers at these frequencies are 1, $m^2/4$, $m^2/4$. Since we assumed linear relation between instantaneous value of modul<u>ating</u> wave and amplitude of modul<u>ated</u> wave, see Fig. 2, no harmonics were generated in process of modulation.

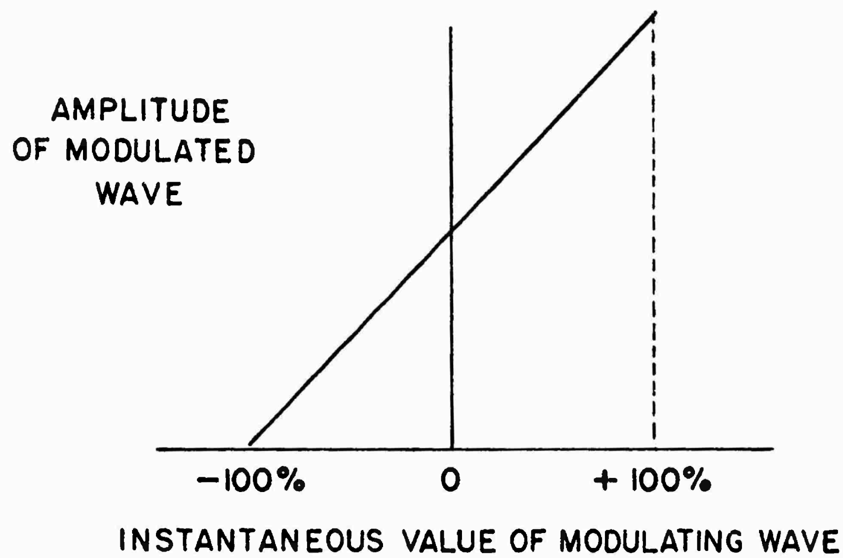



FIG. 2 LINEAR MODULATION

The frequencies present are then shown in Fig. 3.

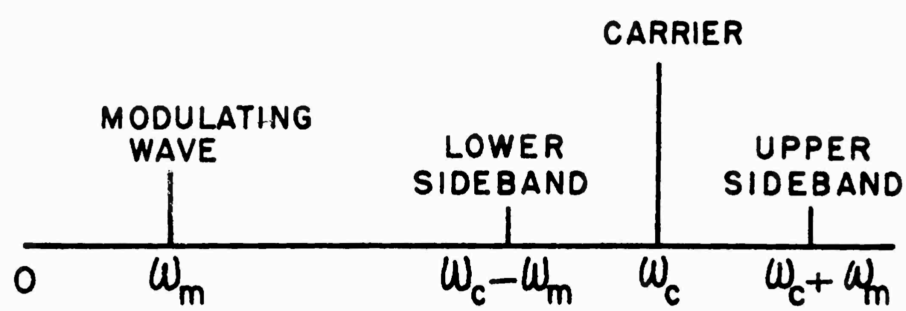
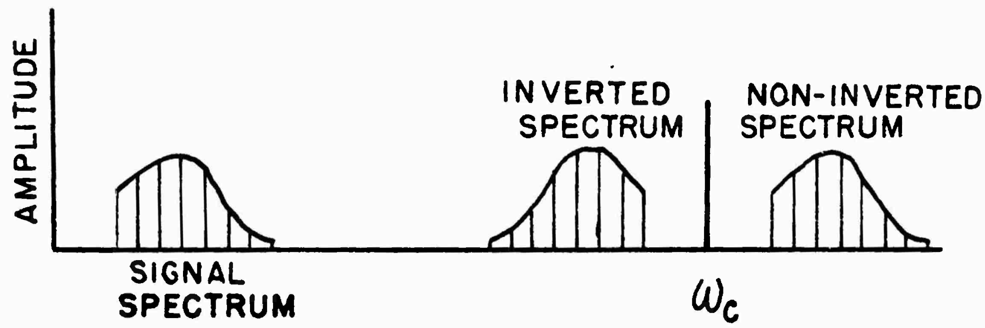



FIG. 3 SPECTRUM OF MODULATION PROCESS

If the modulation signal is not a single sinusoid then we have a certain spectrum of frequencies as shown in Fig. 4. This spectrum is inverted for the lower sideband and not inverted for the upper sideband.

FIG. 4 INVERTED AND NON-INVERTED SPECTRA




Example-Square Law Modulator

If we assume a square law diode modulator, Fig. 5, which does not have linear relation between instantaneous value of modulating wave and amplitude of modulated wave, then harmonics are also produced. This device is actually a nonlinear resistance since its characteristic is specified in the voltage-current plane.

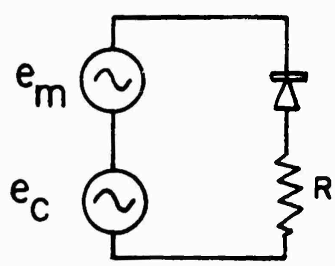



FIG. 5 SQUARE LAW MODULATOR CIRCUIT

Assume $e_m = E_m \cos \omega_m t$

$$e_c = E_c \cos \omega_c t$$

$$e_i = E_c \cos \omega_c t + E_m \cos \omega_m t$$

If $i_p = a_1 e_i + a_2 e_i^2$ i.e., a linear term and a square term,

$$i_p = a_1 E_c \cos \omega_c t + a_1 E_m \cos \omega_m t + a_2 (E_c \cos \omega_c t + E_m \cos \omega_m t)^2$$

$$i_p = a_1 E_c \cos \omega_c t + a_1 E_m \cos \omega_m t + a_2 E_c^2 \cos^2 \omega_c t$$

$$+ a_2 E_m^2 \cos^2 \omega_m t + 2a_2 E_c E_m \cos \omega_c t \cos \omega_m t$$

$$i_p = a_1 E_c \cos \omega_c t + a_1 E_m \cos \omega_m t + a_2 E_c^2 \left[ \frac{1 + \cos 2 \omega_c t}{2} \right]$$

$$+ a_2 E_m^2 \left[ \frac{1 + \cos 2 \omega_m t}{2} \right] + 2 a_2 E_c E_m \left[ \frac{1}{2} \cos (\omega_c + \omega_m) t + \frac{1}{2} \cos (\omega_c - \omega_m) t \right]$$

$$i_p = a_1 E_c \cos \omega_c t + a_1 E_m \cos \omega_m t + \frac{a_2 E_c^2}{2} + \frac{a_2 E_c^2}{2} \cos 2 \omega_c t$$

$$+ \frac{a_2 E_m^2}{2} + \frac{a_2 E_m^2}{2} \cos 2 \omega_m t + a_2 E_c E_m \cos (\omega_c + \omega_m) t + a_2 E_c E_m \cos (\omega_c - \omega_m) t$$

$$i_p = \frac{a_2}{2} (E_c^2 + E_m^2) \text{ - - - - - - - - - - DC} \qquad \text{(Eq. 2)}$$

$+ a_1 E_m \cos \omega_m t$ - - - - - - - - - - Modulating Freq. (from the linear term)

$+ \frac{a_2}{2} E_m^2 \cos 2 \omega_m t$ - - - - - - - - - 2nd harmonic of modulating freq.

$+ (a_2 E_c E_m) \cos (\omega_c - \omega_m) t$ - - - - lower sideband

$+ a_1 E_c \cos \omega_c t$ - - - - - - Carrier (from the linear term)

$+ a_2 E_c E_m \cos (\omega_c + \omega_m) t$ - - - - - upper sideband

$+ \frac{a_2}{2} E_c^2 \cos 2 \omega_c t$ - - - - - 2nd harmonic of carrier

We now see a d.c. term and harmonic terms appearing due to the non-linearity between the instantaneous value of the modulating wave and the amplitude of the modulated wave. If we calculate powers at these frequencies - -

$$\text{power} = i_p^2 r = \left[ \frac{a_2}{2} (E_c^2 + E_m^2) + a_1 E_m \cos \omega_m t + \frac{a_2 E_m^2}{2} \cos \omega_m t + A_2 E_c E_m \cos (\omega_c - \omega_m) t + a_1 E_c \cos \omega_c t + a_2 E_c E_m \cos (\omega_c + \omega_m) t + \frac{a_2}{2} E_c^2 \cos 2 \omega_c t \right]^2 r$$

Squaring $i_p$ and using trig identity $\cos^2 \sigma = \frac{1 + \cos 2 \sigma}{2}$ and recalling that cos terms average to zero.

$$\text{Average power} = I^2 r = \frac{a_2^2}{4}(E_c^2 + E_m^2)^2 r \qquad \text{DC}$$

$$+ \frac{a_1^2 E_m^2}{2} r \qquad \text{modulator freq.}$$

$$+ \frac{a_2^2 E_m^4}{8} r \qquad \text{modulator 2f}$$

$$+ \frac{a_2^2 E_c^2 E_m^2}{2} r \qquad \text{lsb}$$

$$+ \frac{a_1^2 E_c^2}{2} r \qquad \text{carrier}$$

$$+ \frac{a_2^2 E_c^2 E_m^2}{2} r \qquad \text{usb}$$

$$+ \frac{a_2^2 E_c^4}{4} r \qquad \text{carrier 2f}$$

Therefore, real power is present at all of these frequencies.

The presence of these and higher frequencies corresponding to higher harmonics and higher order modulation products require that special consideration be given to the parametric amplifier (having alternating local power). Essentially there are multiple output parts in these amplifiers and under certain conditions multiple input ports. Frequency translation is inherent in the amplification process and, as we shall see in the next section, there are restraints on the conversion of power between frequencies.

## CONSERVATION THEOREMS

Amplifiers excited by alternating current local power are designated by the generic term parametric amplifiers. These amplifiers depend for their operation on the interaction of power at various frequencies in a non-linear or time varying reactance. The agent may be quite varied, such as the domain wall motion or $\vec{s}$tate of magnetization in a low frequency magnetic amplifier, the precessing $\vec{M}$ vector (gyromagnetic state) of a high frequency magnetic amplifier or the charge on a non-linear capacitor. However, in every case the mechanism whereby energy is coupled from the alternating power source to the alternating signal is associated with a non-linearity that can be described as a non-linear or time varying L, C, or M.

The relationship between the powers at the various frequencies was first derived for the non-linear capacitance and by analogy for the non-linear inductance by Manley and Rowe[3]. The frequency-power formulas of

Manley and Rowe form a set of conservation principles that apply to non-linear or time varying energy-storage elements when used as frequency converters. They are conservation principles in the sense that they describe fundamental relations more or less independent of the details of excitation or of the device used. They are

$$\sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} \frac{m P_{m,n}}{m\omega_1 + n\omega_0} = 0 \qquad \text{(Eq. 3a)}$$

$$\sum_{m=-\infty}^{+\infty} \sum_{n=0}^{+\infty} \frac{n P_{m,n}}{m\omega_1 + n\omega_0} = 0 \qquad \text{(Eq. 3b)}$$

where $P_{m,n}$ is power input at frequency $m\omega_1 + n\omega_0$. These relations constrain the sums of real powers at various frequencies to be zero in a non-linear reactance. They say that because of the non-linearity, net power into the system at one frequency is related to net power out at other frequencies even though the total power $\sum\sum_{mn} P_{mn}$ vanishes.

The restraints on frequencies, that is, the statement that only certain frequencies will have real power associated with them, infers perfectly resistive loads for wanted frequencies and perfectly reactive loads for unwanted frequencies. In addition, to state that power enters or leaves the system at a certain frequency infers some impedance matching conditions. Other than this, the relations are general for all parametric devices of the non-linear reactance type regardless of the details of structure. When distributed media are considered later in this report slight modifications of the formulas will be made.

By conservation of energy the sum of all $P_{m,n}$ must vanish; this is predicted from these equations by multiplying the first by $\omega_1$ and the second by $\omega_0$ and adding. The Manley-Rowe formulas thus constitute the law of conservation of energy and one additional relation[5]. The additional relation is a constraint caused by the fact that $\omega_1$ and $\omega_0$ can be specified independently.

These formulas have been generalized. Haus[4] has discussed the electromagnetic field in a non-linear, anisotropic, inhomogeneous, stationary, lossless medium and found that the Manley-Rowe equations hold in a form that allows the power to be evaluated at the boundary of the system in the form of Poyntings' vector, $\vec{E} \times \vec{H}$. He also discusses a gyromagnetic medium for the signal and generated sideband amplitudes much smaller than the pump.

Penfield[5] has shown that these equations are obeyed by many physical systems. He shows that losslessness is neither a necessary nor sufficient

condition. Systems which are "characterized by a (possible explicitly time-varying) energy state function, such as the Lagrangian, Hamiltonian, internal energy, or co-energy, obey the formulas. Distributed systems described by Hamilton's principle obey the formulas in such a way that the powers can be evaluated at the boundaries of the systems. The formulas thus relate quantities of engineering interest, the powers through the ports. Such distributed systems include non-linear electromagnetic media, ferrites, electron beams, and acoustic fields."

Many energy conversion devices, such as the induction motor obey the formulas. Interest has been expressed in their application to the non-linear theory of hydrodynamic and magnetohydrodynamic stability.

Duinker[6] and Sturrock[7] have both concluded that the formulas hold for any system that obeys Hamilton's canonical equations of motion.

Conservation principles are also deducible for the reactive power relations in non-linear reactances[8]. Similarly real power[9] and reactive power relations are obtainable for non-linear resistances. The four sets of relations are summarized in Table I for systems where power is permitted to flow at three frequencies only and in Table II for systems where power is permitted to flow at four frequencies[8].

All of these frequency power relations have been generalized and extended by Penfield[8]. In particular, he formulated these conservation theorems without constraints on the frequencies. In this form when many frequencies are present they are useful for checking analytic results and generally defining the maximum limits of the system. However, when only a few frequencies are present, particularly the case where modulation products are considered, the formulas are simple and ratios of powers can be calculated.

We are primarily concerned at this point with the Manley-Rowe equations for real power in non-linear reactances. Several derivations[3,8,10] have been made of these equations. We will follow Salzberg[10].

Consider non-linear capacitance modulator, where the charge vs voltage is non-linear, $f_1$ is incommensurable with $f_2$, $Z_1$, $Z_2$ are significant only at the respective source frequencies, $Z_L$ is significant only at the combination frequency $f_3 = mf_2 \pm nf_1$ $m,n = 1,2,3...$

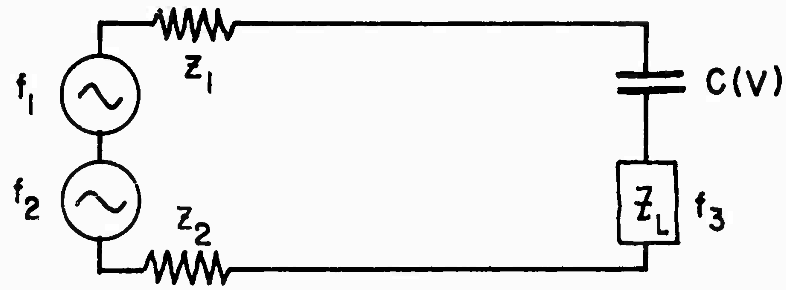



FIG. 6 NONLINEAR CAPACITANCE MODULATOR

The voltage applied to the non-linear C is the sum of load voltage and two source voltages minus corresponding internal source impedance voltage drops... resultant charge is periodic and representable by d.c. term and sinusoidal terms in fundamental and harmonics of $f_1$ and $f_2$ and all sum and difference combinations of $f_1$ and $f_2$ (amplitudes and phases of these are functions of specific non-linear device and amplitude and phase of the 3 voltages but not of frequencies). This can easily be seen if the non-linear characteristic is represented by a power series in voltage. The common steady-state current is the time derivative of the charge and therefore representable by a similar series. The amplitudes of the current terms are proportional to the product of frequency and amplitude of corresponding terms in charge.

In a lossless non-linear capacitance the conservation of energy says that the sum of load power $P_3$, and $P_1$ and $P_2$ be zero.

$$P_1 + P_2 + P_3 = 0$$

Average power ... given by d.c. term in product of volt and current (product of voltage and current of like frequency)

$$P = \frac{VI}{2} \cos \phi \qquad V, I, Q, \text{ at corresponding terms of } f$$

$$= f \quad (\pi V Q \cos \phi) \qquad \phi = \text{phase angle}$$

$$= f w \qquad w = \text{energy/cycle (like Nh in Weiss)}$$

external energy/cycle supplied by each source and for load are

$$w_1 = (\pi V_1 Q_1 \cos \phi_1) = P_1/f_1$$

$$w_2 = (\pi V_2 Q_2 \cos \phi_2) = P_2/f_2$$

$$w_3 = (\pi V_3 Q_3 \cos \phi_3) = P_3/f_3$$

$$f_3 = mf_2 \pm nf_1$$

combining

$$f_1 (\omega_1 \pm n\omega_3) + f_2 (\omega_2 + m\omega_3) = 0$$

Load voltage and current (or charge) are connected by load circuit impedance ( a fctn of $f_3$)
$\therefore$ Charge amplitudes and phases in $f_3 = mf_2 \pm nf_1$ are functions of $f_3$ since $f_3$ is linear combination of $f_1$ and $f_2$ and not their ratio. Then the only admissible solution is set

$$\omega_1 \pm n\omega_3 = 0$$

$$\omega_2 + m\omega_3 = 0$$

Rewriting,

$$-\frac{P_3}{mf_2 \pm nf_1} = \frac{P_2}{mf_2} = \pm \frac{P_1}{nf_1} \qquad \text{(eq. 4)}$$

These are the Manley-Rowe equations for a load circuit responsive only to the sum or difference frequency. The negative sign indicates that the load absorbs power.

Application of Manley-Rowe Formulas

For the ideal non-linear reactance the sum of input powers must equal the total power output at the modulation frequencies. These are $m\omega_p + n\omega_s$ where m and n are positive or negative integers. If the total power is zero then the input frequencies of $\omega_s$ and $\omega_p$ are included.

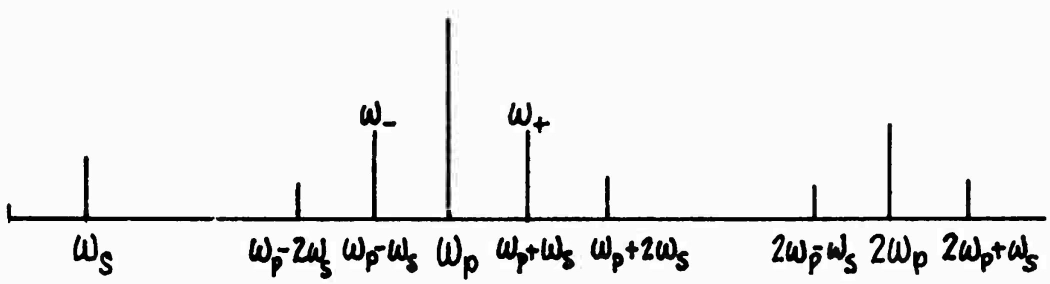



FIG. 7 HIGHER ORDER MODULATION PRODUCTS

If, for example, all outputs were reactively terminated except one, then all input power would necessarily appear at this frequency.

If we consider only the "first order" modulation products $\omega_s$ $\omega_p$ $\omega_+$ $\omega_-$,

neglecting $\omega_p - 2\omega_s$, $\omega_p + 2\omega_s$, $2\omega_p - \omega_s$, $2\omega_p$, $2\omega_p + \omega_s$ as shown in Fig. 7 and higher orders, then the summations in the Manley-Rowe equations are as follows:

$$\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty} \frac{n\ P_{mn}}{m\omega_p + n\omega_s} = 0 \qquad\qquad \sum_{m=1}^{+\infty}\sum_{n=-\infty}^{+\infty} \frac{m\ P_{mn}}{m\omega_p + n\omega_s} = 0$$

| | | | |
|---|---|---|---|
| $n = 1$, $m = 0$ | $\frac{P_{01}}{\omega_s}$ | $m = 1$, $n = 0$ | $\frac{P_{10}}{\omega_p}$ |
| $n = 1$, $m = -1$ | $\frac{P_{-11}}{-(\omega_p-\omega_s)}$ | $m = 1$, $n = 1$ | $\frac{P_{11}}{\omega_p + \omega_s}$ |
| $n = 1$, $m = 1$ | $\frac{P_{11}}{\omega_p + \omega_s}$ | $m = 1$, $n = -1$ | $\frac{P_{1-1}}{\omega_p - \omega_s}$ |

$$\frac{P_{01}}{\omega_s} + \frac{P_{-11}}{-\omega_p + \omega_s} + \frac{P_{11}}{\omega_p + \omega_s} = 0 \qquad\qquad \frac{P_{10}}{\omega_p} + \frac{P_{11}}{\omega_p + \omega_s} + \frac{P_{1-1}}{\omega_p - \omega_s} = 0$$

$$\frac{P_{01}}{\omega_s} + \frac{P_{11}}{\omega_+} + \frac{P_{-11}}{-\omega_p + \omega_s} = 0 \qquad\qquad \frac{P_{10}}{\omega_p} + \frac{P_{11}}{\omega_+} + \frac{P_{1-1}}{\omega_-} = 0$$

$$\frac{P_{01}}{\omega_s} + \frac{P_{11}}{\omega_+} - \frac{P_-}{\omega_-} = 0 \qquad\qquad \frac{P_{10}}{\omega_p} + \frac{P_{11}}{\omega_+} + \frac{P_-}{\omega_-} = 0$$

(eq. 5a) (eg. 5b)

These are then the Manley-Rowe equations where signal, pump and both side-band frequencies are allowed to carry power. Table II shows these relations along with the other power-frequency relations for a system where four frequencies are present.

If the only frequencies permitted to have real power were the signal, pump and sum frequency (or upper sideband) then the power frequency formulas reduce to (see Fig. 8)

$$\frac{P_p}{\omega_p} + \frac{P_+}{\omega_+} = 0 \qquad \text{(eq. 6a)}$$

## TABLE I

### POWER FREQUENCY RELATIONS FOR SYSTEMS WITH THREE FREQUENCIES

$(\omega_s, \omega_p, \omega_-)$

| | NON-LINEAR | |
|---|---|---|
| | Reactance | Resistance |
| Real Power | I<br>$\frac{P_s}{\omega_s} = \frac{P_-}{\omega_-} = \frac{P_p}{\omega_p}$<br>These are the Manley-Rowe equations. | III<br>$P_s + P_- \geq 0$<br>$P_s + P_p \geq 0$<br>$P_p + P_- \geq 0$ |
| Reactive Power | IV<br>$\frac{Q_s}{\omega_s} + \frac{Q_-}{\omega_-} \geq 0$<br>$\frac{Q_s}{\omega_s} + \frac{Q_p}{\omega_p} \geq 0$<br>$\frac{Q_p}{\omega_p} + \frac{Q_-}{\omega_-} \geq 0$<br>$\geq$ for L<br>$\geq$ for C | II<br>$Q_s = Q_- = -Q_p$ |

$$\frac{P_s}{\omega_s} + \frac{P_+}{\omega_+} = 0 \qquad \text{(eq. 6b)}$$

where P is power flowing into the non-linear reactor at the stated frequency.

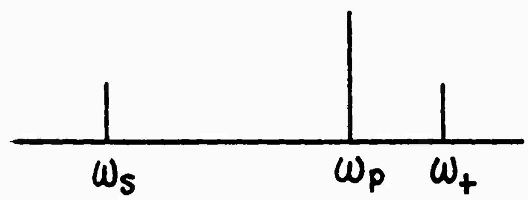



FIG. 8 UPPER SIDEBAND CASE

These and the other power-frequency relations for a three frequency system are shown in Table I. As an upconverting amplifier $P_p$ and $P_s$ are positive and $P_+$ is negative representing useful power output. For a demodulator (down converter) $\omega_+$ is positive and $\omega_s$ and $\omega_p$ negative. The maximum power gain

$$G_p \quad (S \rightarrow +) = \frac{\omega_+}{\omega_s} = \text{non-inverting modulator power gain}$$

$$G_p \quad (+ \rightarrow S) = \frac{\omega_s}{\omega_+} = \text{non-inverting demodulator power gain}$$

Both sources supply power to the load. The power gain of the modulator (up-converter) is the ratio of frequencies and can be high. The gain of the demodulator (down-converter) is the reciprocal and is always a loss, the energy being wasted in the pump circuit. Since these power gains are positive the device is stable since power cannot be delivered to purely passive terminations at both of the signal terminals ($\omega_s$ and $\omega_+$) simultaneously with only the local oscillator applied to the non-linear reactor. This is the theoretical stable gain obtained by a lossless dielectric or magnetic amplifier.

If the only frequencies permitted to carry real power are the signal, pump and difference frequency (lower sideband) then the power frequency formulas reduce to (see Table I and Fig. 9)

$$\frac{P_p}{\omega_p} + \frac{P_-}{\omega_-} = 0 \qquad \text{(Eq. 7a)}$$

$$\frac{P_s}{\omega_s} - \frac{P_-}{\omega_-} = 0 \qquad \text{(Eq. 7b)}$$

These are inverting modulators and demodulators.

TABLE II

POWER FREQUENCY RELATIONS FOR SYSTEMS WITH FOUR FREQUENCIES
$(\omega_s, \omega_p, \omega_-, \omega_+)$

| | NON-LINEAR | |
|---|---|---|
| | Reactance | Resistance |
| Real Power | I<br>$\frac{P_s}{\omega_s} - \frac{P_-}{\omega_-} + \frac{P_+}{\omega_+} = 0$<br>$\frac{P_p}{\omega_p} + \frac{P_-}{\omega_-} + \frac{P_+}{\omega_+} = 0$<br>These are the Manley-Rowe equations. | III<br>$P_s + P_- + P_+ \geq 0$<br>$P_p + P_- + P_+ \geq 0$<br>$P_s + P_p + 4P_- \geq 0$<br>$P_s + P_p + 4P_+ \geq 0$ |
| Reactive Power | IV<br>$\frac{Q_s}{\omega_s} + \frac{Q_-}{\omega_-} + \frac{Q_+}{\omega_+} \geq 0 \ (\leq)$<br>$\frac{Q_p}{\omega_p} + \frac{Q_-}{\omega_-} + \frac{Q_+}{\omega_+} \geq 0 \ (\leq)$<br>$\frac{Q_s}{\omega_s} + \frac{Q_p}{\omega_p} + \frac{4Q_-}{\omega_-} \geq 0 \ (\leq)$<br>$\frac{Q_s}{\omega_s} + \frac{Q_p}{\omega_p} + \frac{4Q_+}{\omega_+} \geq 0 \ (\leq)$<br>For L use $\geq$<br>For C use $(\leq)$ | II<br>$Q_s - Q_- + Q_+ = 0$<br>$Q_p + Q_- + Q_+ = 0$ |

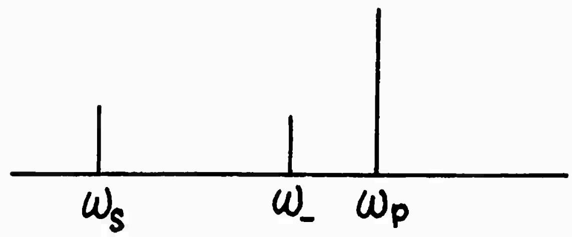



FIG. 9 LOWER SIDEBAND CASE

The power gains are:

$$G_p \quad (s \rightarrow -) = \frac{-\omega_-}{\omega_s} \quad = \text{inverting modulator power gain}$$

$$G_p \quad (- \rightarrow s) = \frac{-\omega_s}{\omega_-} \quad = \text{inverting demodulator power gain}$$

For both modulator and demodulator $P_p$ is positive $P_s$ and $P_-$ negative. The input power from the pump at $\omega_p$ flows out of the non-linear reactor at the two signal frequencies, most of it appearing at the higher frequency $\omega_-$ only a small amount appearing at the lower frequency $\omega_s$. These devices are potentially unstable and under certain conditions will oscillate, power emerging from both $\omega_s$ and $\omega_-$ ports into passive terminations with only pump applied. This is the basis for the negative resistance amplifier exemplified by the multi-level Maser amplifiers and the Suhl magnetic amplifier. In these amplifiers a negative resistance appears in the signal circuit. If the magnitude of this shunt negative resistance is greater than the effective shunt resistance of the signal circuit the signal voltage is augmented. This is amplification. If the shunt negative resistance is less than the effective shunt resistance of the signal circuit, oscillation can result. This potentially unstable device is a regenerative amplifier, that is, a positive feedback is operating. We will, in principle, be able to determine whether or not positive feedback is operating in microwave amplifier employing three frequencies by examining the terminations of the upper and lower sidebands.

We have spoken of allowing or not allowing real power to exist at a certain frequency. At microwave frequencies real power does not exist, for example in a lossless waveguide which is reactively terminated. Only reactive power exists. Real power exists in a resistive termination or in a cavity at resonance. If we build a device of the non-inverting modulator type (upper sideband upconverter) then we wish to have three cavity resonances or other types of resonances corresponding to the three wanted frequencies $\omega_s$, $\omega_p$, $\omega_+$ .

In principle, a resonance is not actually required at the pump frequency as long as pump energy efficiently enters the system. We will then potentially have a device with stable but limited gain. If we build an inverting modulator type (lower sideband upconverter) then we wish to have three cavity resonances or other types of resonance corresponding to the three wanted frequencies $\omega_s$, $\omega_p$, $\omega_-$. We will then potentially have a device with unlimited gain but somewhat unstable, i.e., a regenerative amplifier. Other interesting properties are obtainable by permitting both sidebands or even certain harmonics to flow. Adams[11] has analyzed four frequency non-linear reactance circuits and reports in addition unlimited gain conditions for signal frequencies higher than the pump when negative input resistance is reflected to the upper sideband $\omega_+$, unlimited gain at $\omega_+$ when the first pump harmonic is permitted to carry power and other interesting results.

We will concern ourselves at this point with the feedback effects responsible for the stable gain properties associated with the upper sideband and the regenerative effects associated with the lower sideband.

## FEEDBACK (REACTION FIELDS)

### Spectral Attenuation and Phase Shift

We have found that there are feedback effects associated with the sidebands about the pump or carrier frequency in the modulation process. Their existence was intimated by the conservation theorems but their nature is as yet undisclosed. To discover this we must now extend our knowledge of the attenuation and phase shift properties over the spectrum.

In the modulation process power is absorbed by the non-linear network at the signal frequency ($\omega_S$) and the pump frequency ($\omega_P$). Power emits from the network at the sidebands $\omega_+$ and $\omega_-$. (Power also emits at $\omega_P$ as in a transmitter but this is subtracted from the $\omega_P$ power input to give a net value applicable to the conservation formulas). Fig. 10 illustrates this.

The question naturally arises as to how a lossless reactance can absorb real power. Consider the waveguide - cavity system shown in Fig. 11. The impedance of the cavity viewed from the $\omega_S$ input waveguide has an impedance $Z = R + jX$. That is, there is a loss or resistive term and a reactive term. With a non-linear reactance in the cavity, power is converted to other frequencies. For proper matching power is not reflected back out of the $\omega_S$ port but emerges from other ports at different frequencies. The cavity does not then appear reactive to $\omega_S$ but resistive. This is an absorption of the real power at frequency $\omega_S$ through the action of a lossless non-linear reactance.

Assume that we have permitted real power to exist at the four frequencies: The signal frequency, the pump frequency, the upper sideband and the lower sideband. Specifying the existence of real power requires that the impedance of this device (or network) have a resistance term at

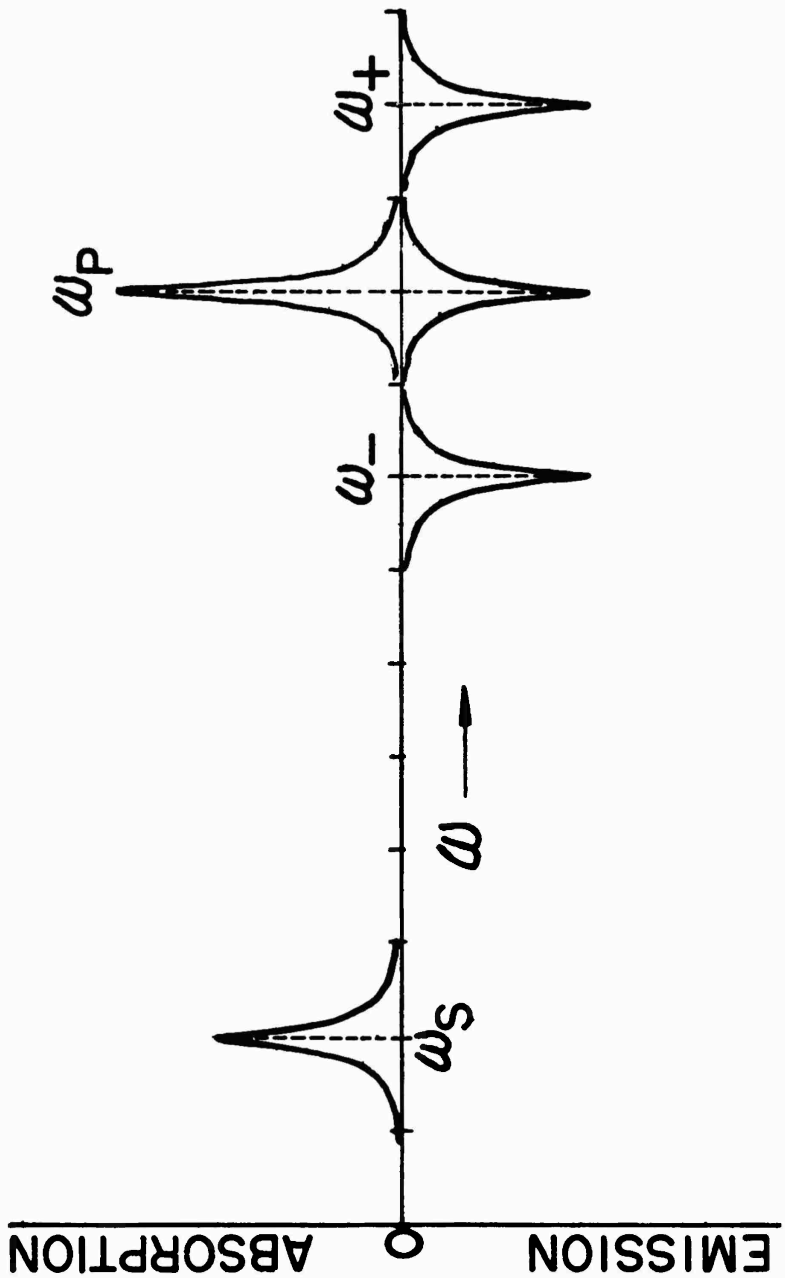



Fig. 10 Absorption and Emission in the Amplitude Modulation Process

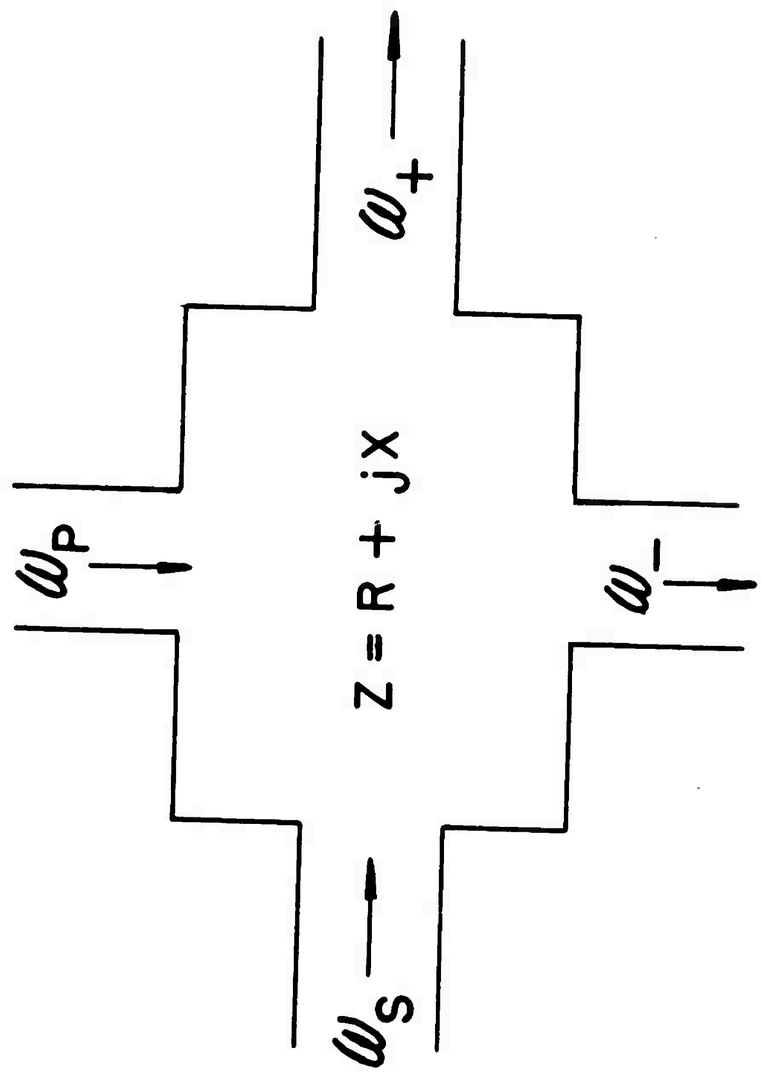



Fig. 11 Absorption of Real Power by Lossless Nonlinear Reactance

the wanted frequency or in resonance terms, it must absorb energy at that frequency. However, absorption (or attenuation) and phase shift are conjugate functions, that is there is a specified relation between the two and one is derivable from the other. These relations are a consequence of functional properties of complex variables and find expression in many forms such as the phase-area theorem[12] of network theory for minimum phase shift networks* and the Kramers-Kronig relations. Simply, the phase shift of a network depends on the rate at which its attenuation varies with frequency.

$$\varphi = \frac{\pi}{12} \left( \frac{dA}{d\mu} \right)_c + \text{a weighting term} \qquad \text{(eq. 8)}$$

where

$\varphi =$ phase shift, radians, at frequency $\nu_c$

$\frac{dA}{d\mu} =$ slope of attenuation curve, db/octave

$\mu = \log_\epsilon \left(\frac{\nu}{\nu_c}\right)$ where $\nu$ is frequency and $\nu_c$ is frequency at which $\varphi$ is desired. For our purposes we can neglect the weighting term. This yields, for example, the familiar $\pi$ radians phase shift associated with an attenuation varying 12 db/octave at the reference frequency. Assume an attenuation function as shown by the dotted curve in Fig. 12(a) which represents typically our absorption curves at the wanted frequencies. The solid line represents the asymtotic approximation to the dotted curves. The resulting phase shift is shown by the dotted line in Fig. 12(b).

For generality we will not indicate what rate of variation of attenuation our structure possesses. Indeed we may not know it. But there will be some phase shift curve associated with it having the form of the dotted line of Fig. 12(b), that is, having first positive phase shift, zero, then negative phase shift as frequency is increased. We can therefore indicate this schematically as in Fig. 13 where with each attenuation curve there is a corresponding phase shift curve.

This attenuation-phase shift spectrum must be superimposed on the absorption-emission spectrum of the modulation process.

## Phase of Feedback

For simplicity of presentation let us now separate the network absorption into the two cases: (1) Where signal, pump, and upperside band frequencies are permitted to carry real power and (2) where signal, pump, and lower sideband frequencies are permitted to carry real power. Case (1) corresponds to equation 6 and will be called the upper sideband case. Case (2) corresponds to equation 7 and is called the lower sideband case.

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* Network where no transverse electro-magnetic wave exists

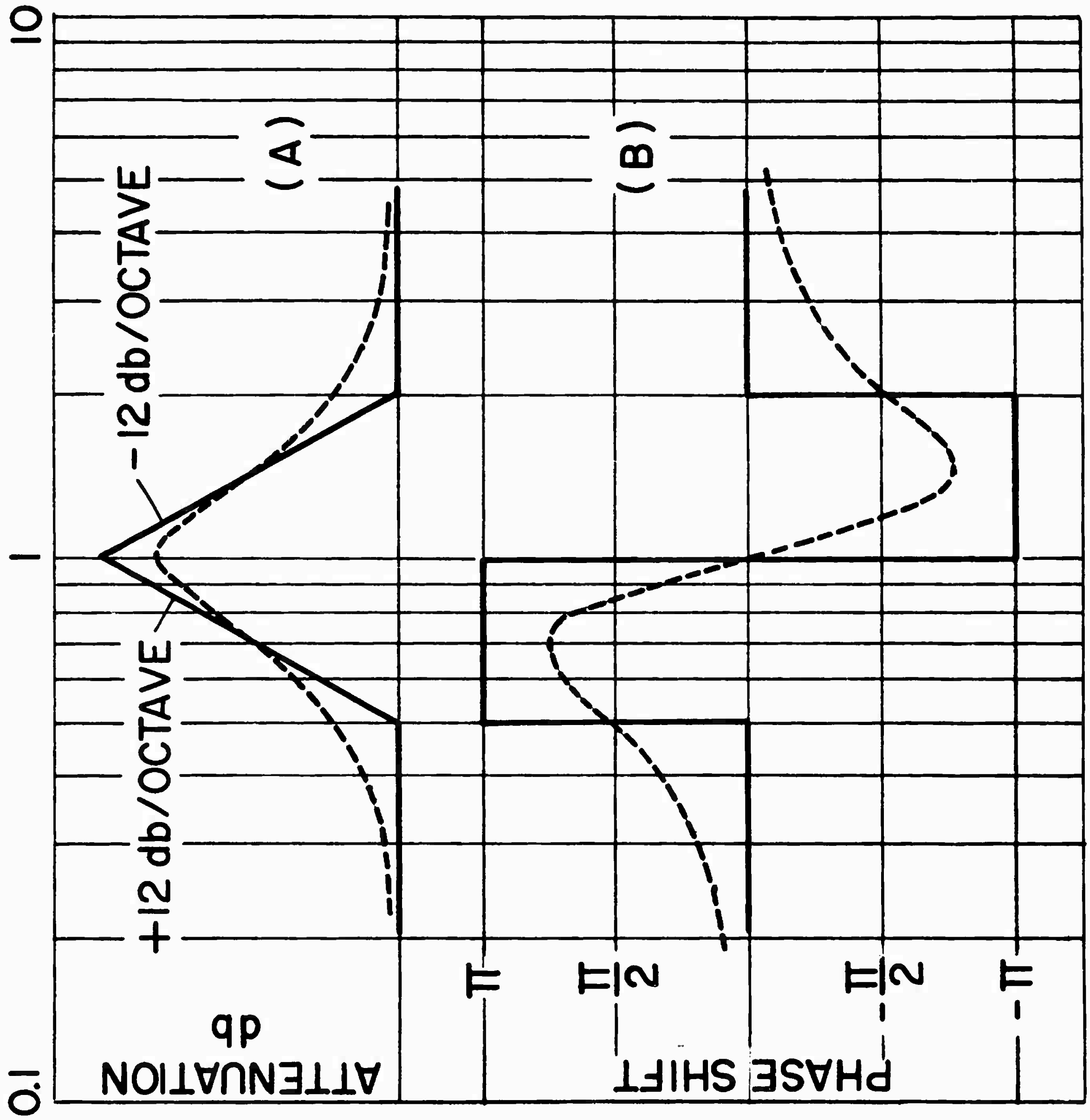


Fig. 12 Attenuation-Phase Shift Spectrum of Resonant Structure

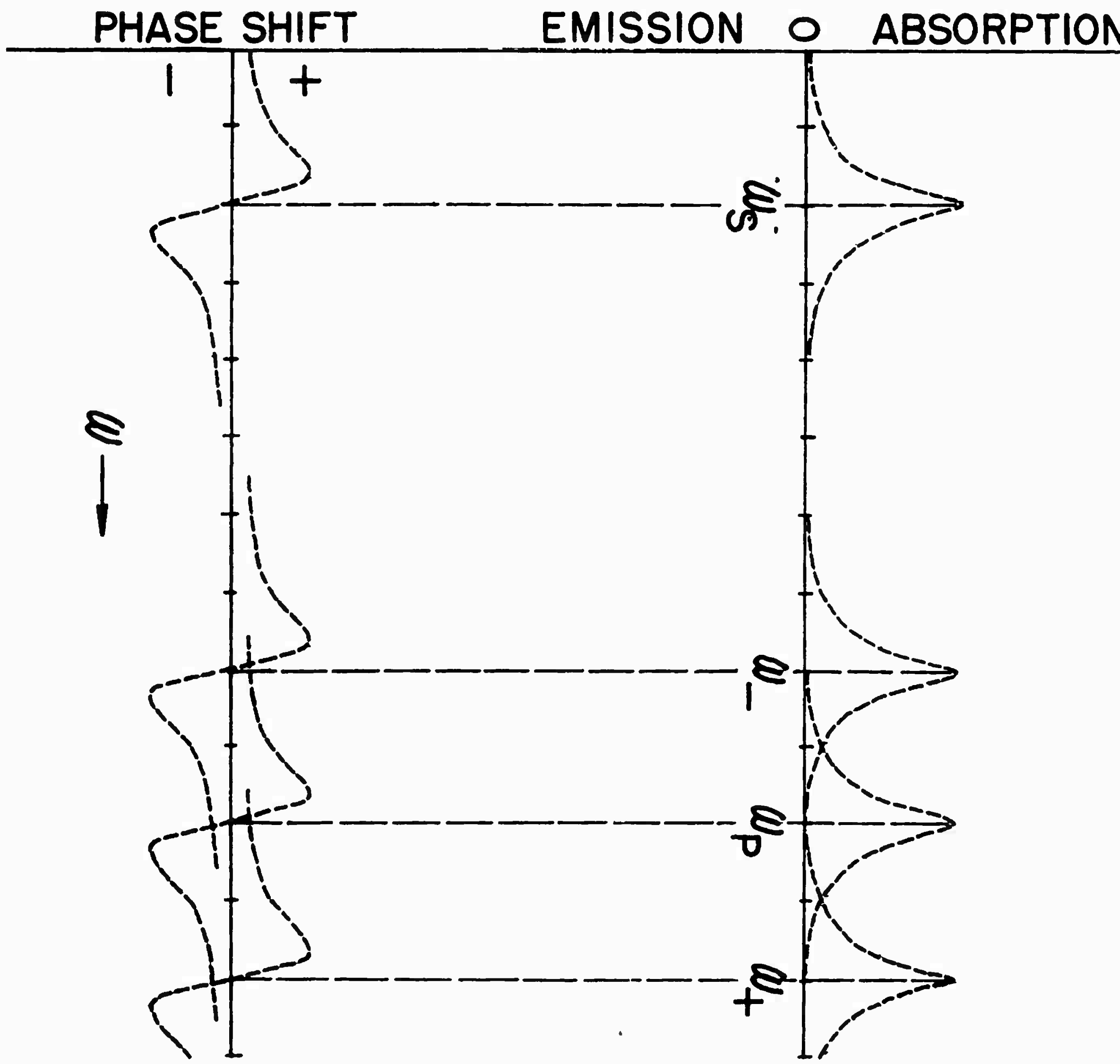


Fig. 13 Attenuation-Phase Shift Spectrum of Multi-Resonant Structure

Lower Sideband Case: Consider first the lower sideband case. There are two steps to the feedback process. In step one the signal power at $\omega_S$ is absorbed by the non-linear network along with pump power at $\omega_P$. The lower sideband, $\omega_-$ is emitted. By definition this power is absorbed in a cavity or other resonant circuit structure coupled to the non-linear structure and represented by the dotted lines of Fig. 14.

The resonant circuit appears resistive with no phase shift. Large fields are produced at $\omega_-$. These "reaction fields"[13] now are presented to the non-linear network as a new signal in step two of the feedback process. As a new signal they generate sidebands about the pump frequency of

$$\omega_P + (\omega_P - \omega_S) = 2\,\omega_P - \omega_S$$

$$\omega_P - (\omega_P - \omega_S) = \omega_S$$

which emit from the non-linear network. The first is reactively terminated by definition. The second is identical in frequency with the original signal. This would be regenerative positive feedback if the phases were opportune and the through gain greater than one.

Consider the phase relations. If the original signal at $\omega_S$ has a phase of $+\theta$ radians, then amplitude modulation theory shows that after step one of the feedback process the phase of the lower sideband is $-\theta$. There has been a phase reversal. This is a well known property of the modulation process and is employed in phase discriminatory devices such as single sideband generators[14]. It can be stated as follows: If the phase of the pump is advanced $\varphi$ degrees, the phase of both sidebands advances $\varphi$ degrees. Conversely if the phase of the signal is advanced $\theta$ degrees the phase of each component of the upper sideband is advanced $\theta$ degrees and the phase of each component of the lower sideband is retarded $\theta$ degrees.

In step two of the feedback process $\omega_-$ is considered to be a new signal producing a new lower sideband at $\omega_S$. Again there is a phase reversal from $-\theta$ back to $+\theta$. The power now emitting at $\omega_S$ is in phase with the original absorbed signal $\omega_S$. In circuit theory language a negative resistance is presented to the $\omega_S$ port. This positive feedback results in regeneration provided the loop gain is greater than one. This last condition can be determined only from the conservation theorems. Equation 7 shows the overall power gain to this device to be infinite so we conclude that the loop gain is greater than one. The process can repeat indefinitely. Since $\omega_S$ and $\omega_-$ are both present in the network then either can be coupled out as useful output. If $\omega_S$ is taken as output a negative resistance amplifier results, if $\omega_-$ is taken a lower sideband upconverter results. Also, either can be injected as the real signal. This is usually called a double channel amplifier.

The phase shift associated with the absorption of the network has not yet entered the discussion since it was assumed that the frequencies fell on the centers of resonance curves.

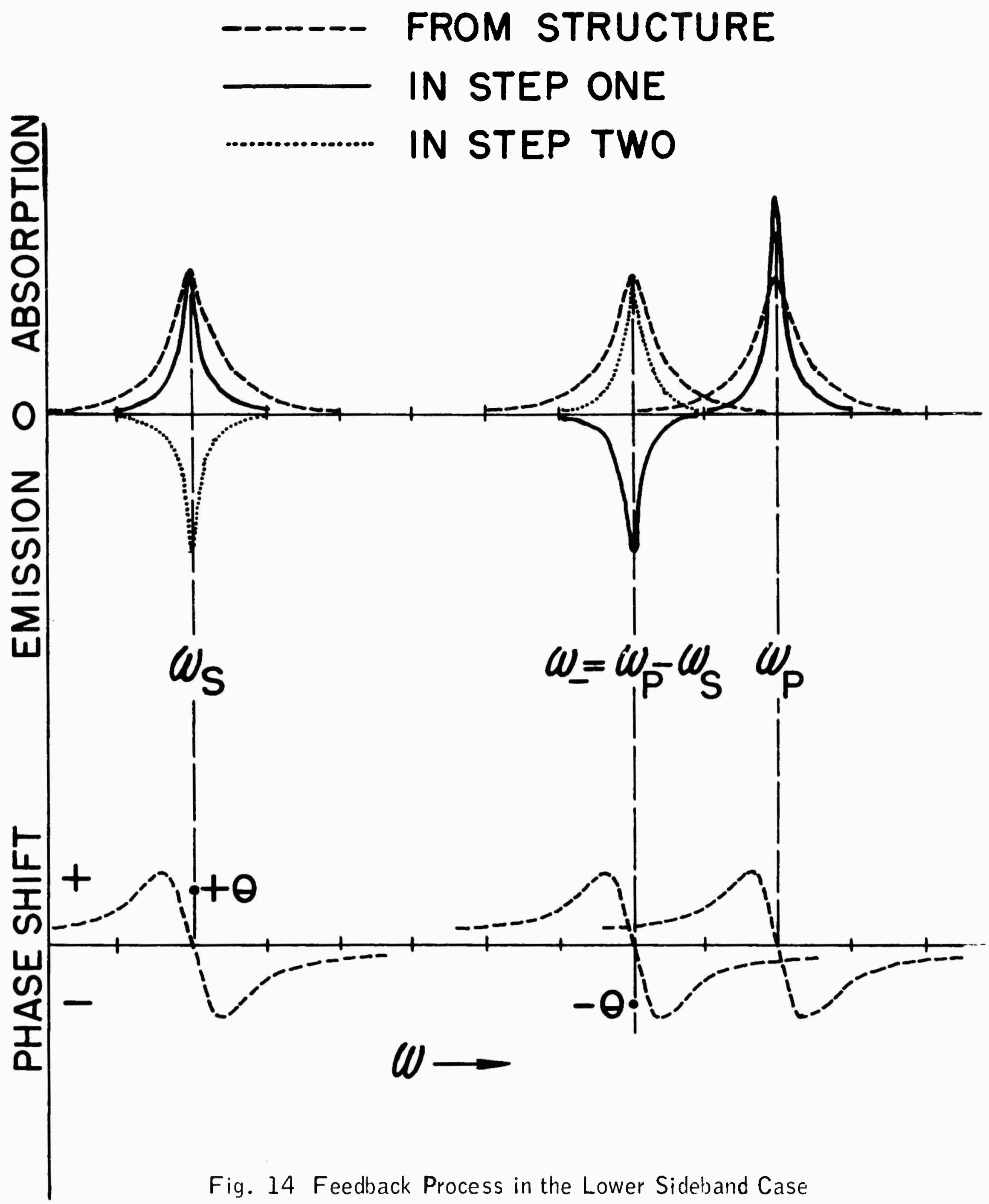


Fig. 14 Feedback Process in the Lower Sideband Case

It is obvious that as the frequencies shift off the resonances additional phase shifts will be introduced from the network, reducing the feedback process and eventually destroying it.

It is intriguing to consider the use of other network non-linearities such as a non-linear resistor. Phase shift could be introduced by operating off-resonance. The power frequency theorems show, however, that the only part of power that is conserved in a non-linear resistance is the reactive part. Since we are interested in the real part of the power it must be shown that the non-linear resistance in combination with the off-resonance reactance of the resonant network provide the necessary real power-frequency conversion for amplification.

Degenerate Case: The regenerative process involved two steps where the phase of the signal was reversed twice. Thus the regenerative process is independent of the phase of the signal. This is not so for one special case, the degenerate case. When the signal frequency $\omega_S = \frac{\omega_P}{2}$, then the signal and lower sideband frequencies are the same. In addition the frequencies are commensurate and relative phase has meaning. Now for a signal at $\omega_S$ with phase $+ \theta$ relative to the pump there emits a lower sideband at $\omega_S$ with phase $-\theta$. Cancelation is immediate for all $\theta \mp 0$ or $2\pi$ relative to the pump. Regeneration can only take place then in the degenerate case for $\theta = 0$ or $2\pi$. That is, the signal must be in phase with the pump which is twice its frequency, or shifted by a phase angle of $2\pi$ (referred to $\omega_P$). The signal is commonly said to be in or out of phase with the pump. These two phase conditions are shown in Fig. 15.

Upper Sideband Case: Again there are two steps to the feedback process. First signal power at $\omega_S$ is absorbed by the network along with pump power at $\omega_P$. An upper sideband is produced at $\omega_+$ and emits from the non-linear element to the resonant circuit. As before, the phase of the signal, $\omega_S$, is $+ \theta$ radians. From modulation theory and reference 14 the phase of $\omega_+$ is $+ \theta$. Being resonant, large fields are produced which react back on the non-linear element. That is, $\omega_+$ now appears as a pump producing sidebands at $\omega_P + \omega_+ = 2\,\omega_P + \omega_S$ and $\omega_P - \omega_+ = \omega_P - \omega_P - \omega_S = -\omega_S$. The first is reactively terminated and is negligible by definition. The second is a frequency equal to the original signal frequency but negative. This is a negative frequency, which is sometimes encountered in analyses in the complex frequency plane. It will be recalled that "the real component of the impedance is an even function of frequency, and the imaginary component is an odd function. In other words, the real component of the impedance at a negative frequency is equal to its value at the corresponding positive frequency, while the imaginary component at a negative frequency is the negative of the imaginary component at the corresponding positive frequency."[15] We can thus plot attenuation and phase shift for $-\omega_S$ as shown in Fig. 16. To verify this consider the effect of lowering the frequency of $\omega_S$ from the midpoint of the absorption curve. This introduces positive phase shift. Reducing frequency on the negative frequency side of the zero line introduces a negative phase shift as it should.

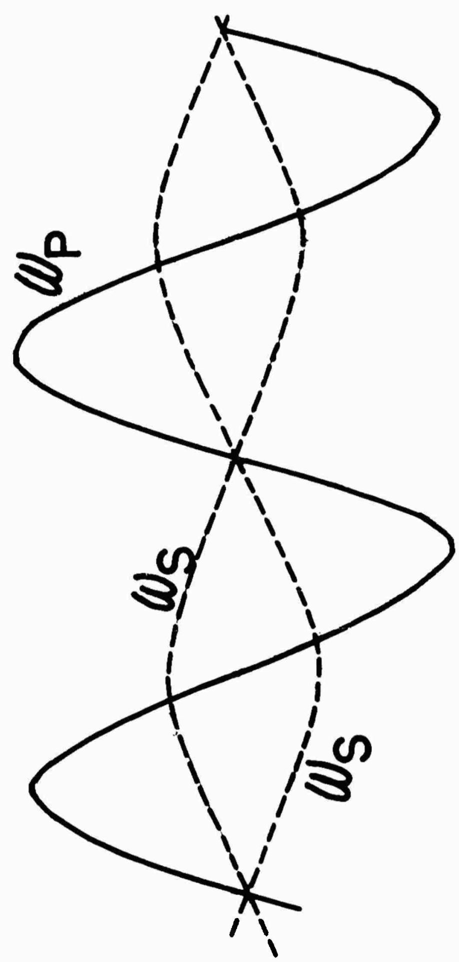


Fig. 15 Phase Relations in the Degenerate Case

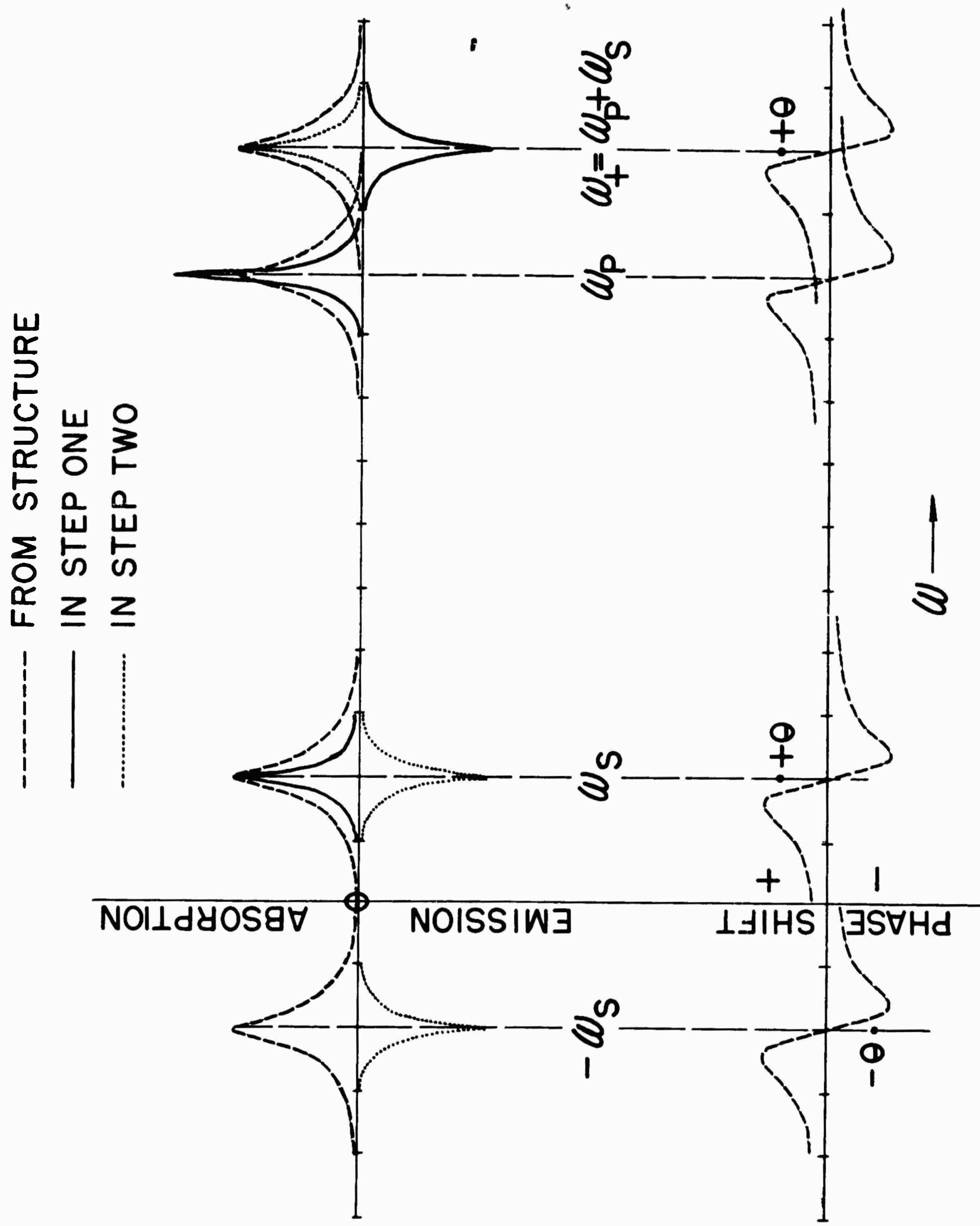


Fig. 16 Feedback Process in the Upper Sideband Case

Now return to step two of our feedback process. The new lower sideband at $-\omega_S$ has a phase shift associated with it of $-\theta$ radians. Recalling symmetry conditions this corresponds to a $+\theta$ phase shift at $+\omega_S$ frequency. Thus the power emitting from the system at $\omega_S$ is in phase with the original signal and this feedback is also positive. This is rather surprising since negative feedback has been suggested as the mechanism limiting the gain of the upper sideband case. Another factor, however, enters the picture, the so-called loop gain of the amplifier feedback system.

Representing the two steps of the feedback process by the Block diagram of Fig. 17, block A is seen to represent step one where signal frequency $\omega_S$ and pump frequency $\omega_P$ produce the upper sideband $\omega_+$. Block B represents step 2 where the upper sideband in the presence of the pump frequency produces $\omega_S$ again. We have seen above that this feedback process is positive. $P_S^!$ and $\omega_S^!$ have their usual meaning of power and frequency and

$$P_+ = A(P_S + P_f)$$

$$P_f = B\ P_+$$

$$P_+ = A(P_S + BP_+) = AP_S + ABP_+$$

$$P_+ = \frac{AP_S}{(1-AB)}$$

$$\frac{P_+}{P_S} = \frac{A}{1-AB}$$

This is the resultant gain for $\omega_S$ input and $\omega_+$ output. The conservation theorems tell us what this gain is without losses.

$$\frac{P_+}{P_S} = \frac{A}{1-AB} = \frac{\omega_+}{\omega_S}$$

Since this is finite, AB is less than 1.

We see that the stable gain properties associated with the upper sideband in the three frequency parametric amplifier and the regenerative effects associated with the lower sideband can be explained in terms of normal feedback theory. As a result the phase independence of the non-degenerate lower sideband case appears logical and consistent. The feedback is also found to be positive for the upper sideband case but the loop gain remains less than unity.

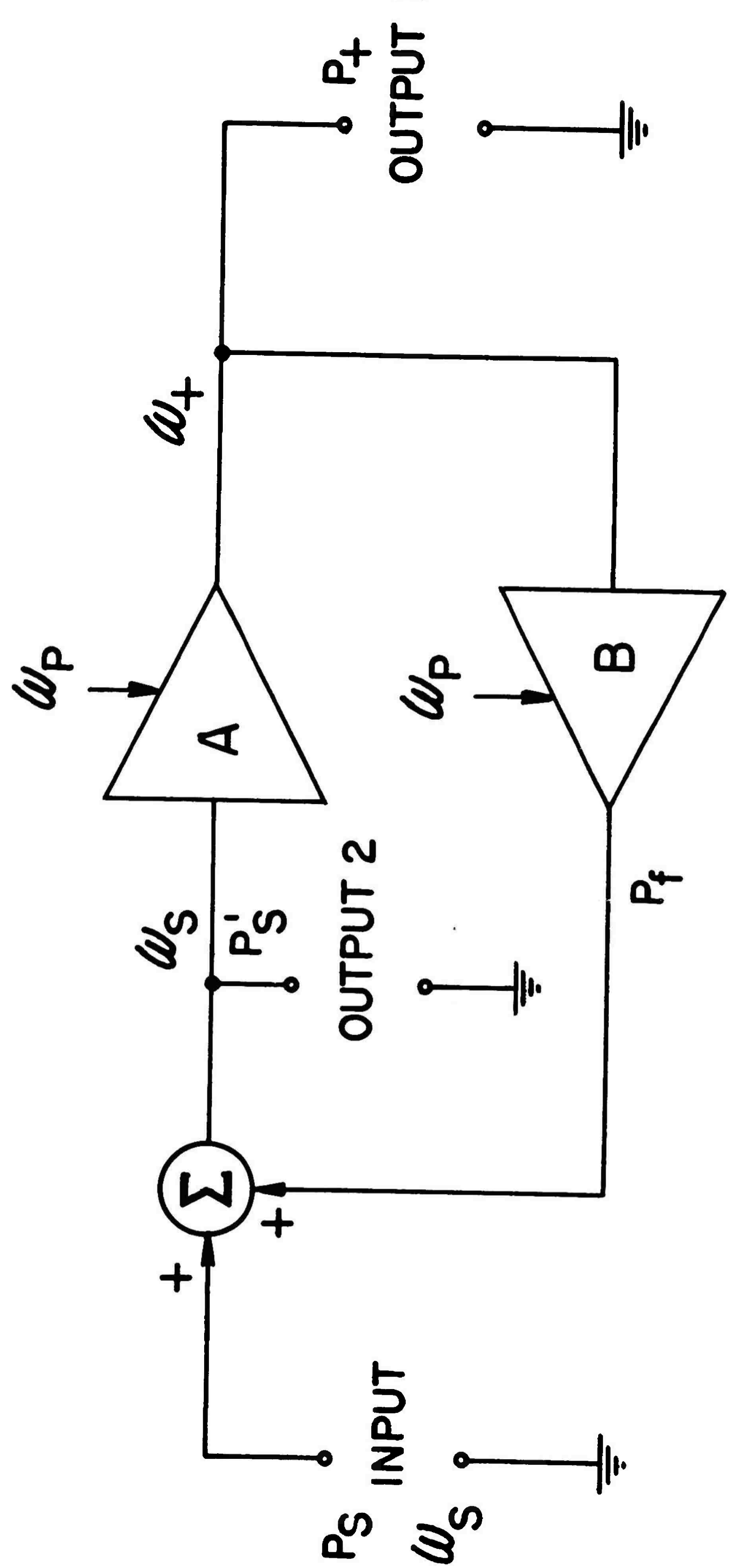


Fig. 17 Block Diagram of Feedback Process in the Upper Sideband Case

## PHYSICAL MECHANISMS OF PUMPING AND COUPLING IN RESONANT SYSTEMS

The basic principles of the regenerative three-frequency parametric amplifier are well known[3]. A time-varying or non-linear reactance when driven or pumped by an alternating source produces an array of harmonics and in the presence of a signal frequency produces an array of modulation products. In the three frequency amplifier the modulation products of particular interest are the upper and lower signal sidebands of the pump frequency. Conservation theorems in the form of the Manley-Rowe relations establish that the restriction of real power to the signal and lower sideband circuits produces regenerative effects. The most common embodiment of this principle is that of a variable reactance, a pump, a signal resonant circuit and an idler or lower sideband resonant circuit.

The degenerate case of this principle results when signal and idler are each one-half the pump frequency and a single resonant circuit supports both.

Detailed circuit analyses of these systems have been made once it is assumed that pumping of the variable reactance has taken place. It is the purpose here to describe the less well understood physical mechanism of the parametric pumping of a resonant circuit and subsequently the coupling between pumped resonant circuits. This can be done using either a mechanical model or an electrical model. The mechanical model, i.e., pendula, will be treated and the analogy drawn to the electrical case.

### Pumping

Consider the pendulum of Fig. 18 oscillating in the plane P. Also in this plane is an electric field E. If the pendulum bob is considered to be positively charged then obviously the motion of the pendulum can be modified[16] by varying the electric field E. If, for example, the field E is turned on when the pendulum is at its two positions of maximum excursion and turned off when the pendulum passes through the equilibrium position, then the magnitude of the pendulum motion will be increased. The pendulum will be "pumped."

With the electric field removed the equation of motion of the pendulum for small amplitudes is

$$\frac{d^2x}{dt^2} + \frac{g_1}{l_1} x = 0 \qquad \text{(eq. 9)}$$

The periodic modulation of the electric field is equivalent to a periodic modulation of the gravity coefficient g and therefore constitutes the "parametric" term in this "parametric" equation.

Another obvious way to pump the pendulum is to vary l. This is one method employed by the monkey swinging from a tree limb or a child in a swing when he wishes to increase the amplitude of swing.

$$\omega_1 = \sqrt{\frac{g_1}{l_1}}$$

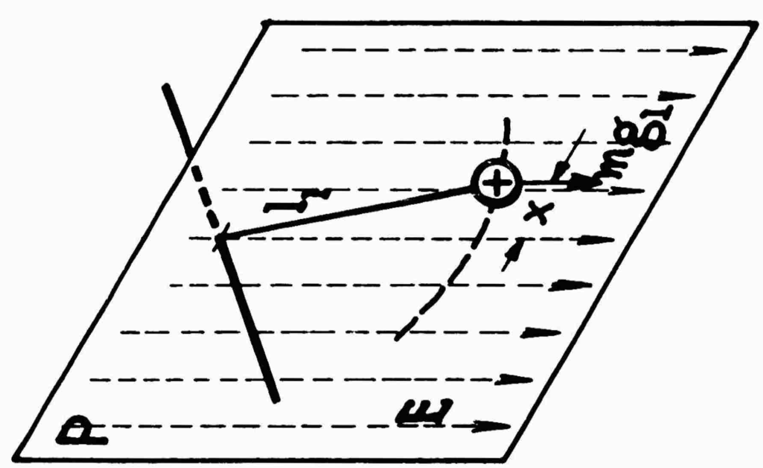

Fig. 18 Parametrically Pumped Pendulum

For our case g is the parametric term and for a sinusoidal variation of the electric field at frequency $\omega_p$ we can write:

$$g_1 = g_o + g_p \cos \omega_p t$$

where $g_o$ = normal gravity field coefficient unperturbed

$g_p \cos \omega_p t$ = apparent perturbation of gravity field coefficient due to electric field

$$\text{Then } \frac{d^2x}{dt^2} + \frac{g_o + g_p \cos \omega_p t}{l} x = o \qquad \text{(Eq. 10)}$$

This equation is of the form of a Mathieu's equation[17]. If the E field variation were an arbitrary periodic function of time then equation 10 would become Hill's equation.

If the substitution is made

$$x = u \qquad \xi = 1/2\ \omega_p t$$

$$\frac{\partial^2 u}{\partial \xi^2} + (\eta + \gamma \cos 2\xi)\, u = o \qquad \text{(Eq. 11)}$$

where $\eta = \dfrac{4\, g_o}{l\omega_p^2}$ $\qquad \gamma = \dfrac{4\, g_p}{l\, \omega_p^2}$

Equation 11 is still Mathieu's equation, a particular case of a linear second order differential equation with periodic coefficients. Floquet's theory applies and the general solution is of the type

$$u = C_1\, A(\xi)\, e^{\mu\xi} + C_2\, A(-\xi)\, e^{-\mu\xi} \qquad \text{(Eq. 12)}$$

Where A $(\xi)$ is a periodic function of $\xi$ with period $\pi$.

With the frequency chosen the complex propagation constant $\mu$ is obtained which may be $\mu = \alpha + i\beta$, which is complex or real indicating attenuated motion. Both $\eta$ and $\gamma$ are proportional to $\dfrac{1}{\omega_p^2}$. Their ratio is constant. Fig. 19 is a plot of these variables and the resultant values of $\mu$.

As indicated by Brillouin[17] the usual solutions sought after are for $\gamma = 0$ where there is no periodic perturbation. A $(\xi)$ is a constant and the solution to equation 12 can be written in terms of cosines or sines.

$$\cos m\xi = 1/2\, (e^{im\xi} + e^{-im\xi})$$

where $C_1 = C_2 = 1/2$ and $\mu_0 = im = i\sqrt{\eta}$ or

$$\sin m\xi = \frac{1}{2i}\left(e^{im\xi} - e^{-im\xi}\right)$$

where $C_1 = C_2 = \frac{1}{2i}$

For $\gamma = 0$ Mathieu functions have been defined as expansions in powers of $\gamma$ using either $\cos m\xi$ or $\sin m\xi$ and where $\mu = im$. Fig. 19 is a plot of the Mathieu functions for integral values of m. These start from the points $\eta = m^2 = 1,4,9,16$, etc., on the $\eta$ axis. Tables of Mathieu functions are available[18,19]. Non-integral values of $\mu$ are indicated in the blank regions of Fig. 19.

## Pass and Stop Bands

The blank regions then are those for which $\mu$ is pure imaginary and no attenuation takes place. The shaded regions correspond to values of $(n,\gamma)$ for which $\mu$ is complex or real and attenuation occurs.

Shaded regions are stop bands and blank regions are pass bands. Consequently, for the pumped pendulum blank regions mean $(\mu = i\beta)$ stable oscillations of constant average amplitude; that is, no energy from the pump source is "stopped" or absorbed by the system and oscillation continue normally. The shaded regions (stop bands) mean for the pumped pendulum that pump energy is absorbed or "stopped" by the system. Since losses are not considered, this means unstable oscillations increasing to infinity.

Methods are available for the determination of the characteristic exponent $\mu$ in the regions of instability. Hayashi[20] following Whittaker[21] has found these solutions for small $|\gamma|$ having the form $u = e^{\mu\xi} A(\xi)$ where $A(\xi)$ to first order is $\sin(n\xi - \sigma)$.

In the first instability region

$$\mu = 0.25\gamma \sin 2\sigma - .0024\, \gamma^3 \sin 2\sigma + \ldots.$$

$$\eta = 1 + 0.5\,\gamma \cos 2\gamma + .004\,\gamma^2 (-16 + 8 \cos 4\sigma)\ldots.$$

where $\gamma$ is a new parameter ranging in value from 0 to $-\frac{\pi}{2}$. Figure 20(a) is a plot from Hayashi of values of $\mu$ and $\sigma$ in the first instability region.

In the second instability region

$$\mu = -0.016\,\gamma^2 \sin 2\sigma + \ldots.$$

$$\eta = 4 - 0.125\,\gamma^2 (0.167 - \sin^2\sigma) + \ldots$$

and Fig. 20 (b) is a plot of these values of $\mu$ and $\sigma$.

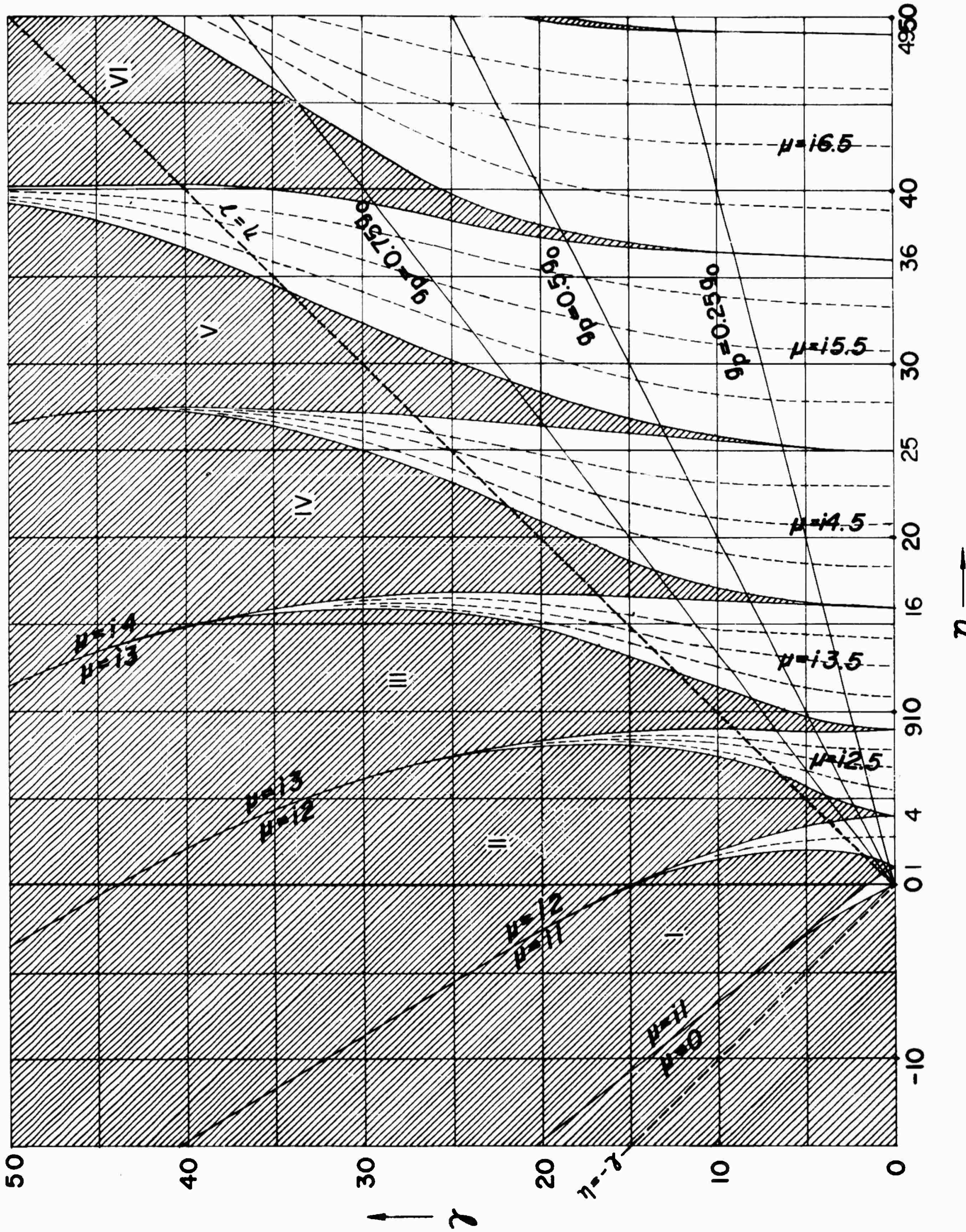


Fig. 19 Stability Chart of Mathieu's Equation

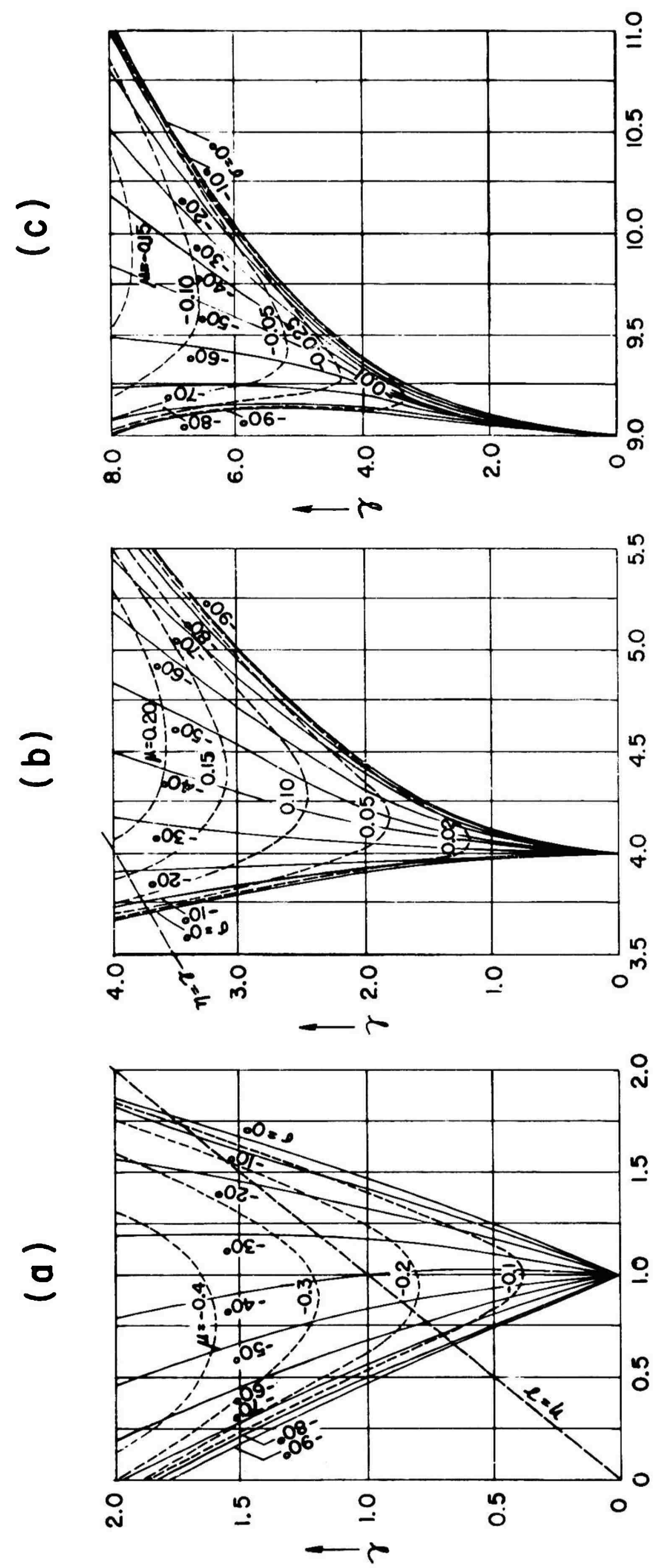


Fig. 20 Instability Regions I, II, III

In the third instability region

$\mu = 0.0003 \sin 2\sigma + \ldots$

$\eta = 9 + .016\,\gamma^2 + .002\,\gamma^3 \cos 2\sigma + \ldots$

and Fig. 20(c) is a plot of these values.

Normally $\eta$ and $\gamma$ are known, $\sigma$ is determined from the second of each of these sets of equations and substituted in the first to give the propagation constant, $\mu$.

In the first and third instability regions and by extention all odd instability regions, $\mu$ is real and negative. There is then a damping term $e^{-\alpha}$ in equation 12, indicating that the system, in this case the pendulum, gives up energy to the pumping source. In the second and all even instability regions $\mu$ is real and positive. There is then an exponentially growing term in equation 12 and the system absorbs energy from the pumping source.

If $-\sigma$ is written for $\sigma$, the value of $\eta$ is unchanged but the sign of $\mu$ is altered. A second independent solution of the form $u = e^{-\mu\xi}(\sin n\xi + \sigma)$ results and the complete solution with two arbitrary constants is

$$u = C_1 e^{\mu\xi} \sin(\eta\xi - \sigma) + C_2\, e^{-\mu\xi} \sin(\eta\xi + \sigma) \qquad \text{(eq. 13)}$$

The situation is now reversed and even numbered instability regions give up energy to the pump while odd instability regions absorb energy from the pump. This will be recognized as the necessary phase condition on pumping to increase or decrease the amplitude of oscillation of the pendulum.

Frequency Relations

The stability map of Fig. 19 then indicates the regions (shaded) where the pendulum can be pumped, i.e., receive net energy from the pump energy source if the pump and oscillating system have the proper phase relation. For the normal pendulum with g directed downward and g and l positive, implying $\eta > \gamma > o$, operation is restricted to the region to the right of and below the line $\eta = \gamma$. In this region stability is the norm and instability (pumping possible) is the exception.

Since $\eta = \dfrac{4\, g_o}{l\, \omega_p^2}$ and $\gamma = \dfrac{4\, g_p}{l\, \omega_p^2}$

in the first instability region ($\eta = 1$)

$$\omega_p^2 = \frac{4\, g_o}{l} = 4\, \omega_o^2 \qquad \text{(eq. 14)}$$

$$\omega_p = 2\, \omega_o$$

This is the familiar requirement of degenerate parametric pumping that the pump frequency be twice the signal (and idler) frequency.

In the second instability region ($\eta = 4$), the pump frequency is the same as the signal frequency $\omega_p = \omega_o$. Subsequent instability regions produce pump frequency and system resonant frequency relations as shown in Table III.

TABLE III

PUMP AND SYSTEM RESONANT FREQUENCY RELATIONS

| Instability Region | (For $\gamma = o$) $\eta$ | $\omega_p$, $\omega_o$ relations for $\sigma = o$ | $\omega_p$, $\omega_o$ relation for $\gamma = \eta$ (Center of Instability Region) | For $\gamma = \eta$ $\eta$ |
|---|---|---|---|---|
| I | 1 | $\omega_p = 2\omega_o$ | $\omega_p = 1.8\ \omega_o$ | 1.2 |
| II | 4 | $\omega_p = \omega_o$ | $\omega_p = 0.89\ \omega_o$ | 5 |
| III | 9 | $\omega_p = 2/3\ \omega_o$ | $\omega_p = 0.62\ \omega_o$ | 10.6 |
| IV | 16 | $\omega_p = 1/2\ \omega_o$ | $\omega_p = .46\ \omega_o$ | 19.2 |
| V | 25 | $\omega_p = 2/5\ \omega_o$ | $\omega_p = .36\ \omega_o$ | 30.7 |
| VI | 36 | $\omega_p = 1/3\ \omega_o$ | | |
| VII | 49 | $\omega_p = 2/7\ \omega_o$ | | |

It will be noted that operation in regions III and higher is equivalent to pumping with a lower frequency than the signal. Operation in higher instability regions becomes increasingly difficult, however, due to the narrow stop band for moderate perturbation of g (small $\gamma$).

Montgomery[22] has obtained the $\omega_p$, $\omega_o$ relations for $\gamma = o$ from other considerations. He has experimentally verified operation in instability region III by obtaining 9 KC oscillations with a 6 KC pump and 15 Mc oscillations with a 10 MC pump using variable capacitance diodes.

Obviously harmonics of the pump frequency will still pump the system for any given instability region. In the first instability region the proper pump frequency is $\omega_p = 2\omega_o$. If the second harmonic of the pump $2\omega_p$ is employed the condition $\omega_p = \omega_o$ results which is the frequency ratio for pumping in instability region 2. Here, however, the pumping frequency is indistinguishable from the system resonant frequency which in an amplifier would be an unsatisfactory condition. For $3\omega_p$ as the "pump" the condition $\omega_p = \frac{2}{3}\,\omega_o$ results which is properly pumping in the third instability region. For $4\ \omega_p$ as the "pump" the condition $\omega_p = \frac{1}{2}\,\omega_o$ results

which is pumping in the fourth instability region where the system frequency is indistinguishable from the pump frequency. For $5\omega_p$ as the "pump", $\omega_p = \frac{2}{5}\omega_o$ corresponding to the fifth instability region. The conclusion then is that when the system is designed for pumping in the first instability region, any odd harmonic of the pump will also pump the system.

If the system is designed for pumping in the third or higher instability regions, even harmonics of the pump again result in operation equivalent to pumping in even numbered instability regions and odd harmonics of the pump result in operation equivalent to pumping in odd numbered instability regions. There is thus an array of frequencies that can properly pump a given resonant system. These are illustrated in Fig. 21 against an arbitrary frequency scale. The pump frequencies denoted by Roman numerals are the first order pump frequencies predicted by the corresponding instability regions of the Mathieu function. The others result from employing a pump frequency which is an odd harmonic of the "pump" frequency predicted by the Mathieu function and thus reduces equivalently to one of the instability regions.

It should be noted that each of these frequencies actually occurs for the $\theta = 0$ case, i.e., along the $\eta$ axis. These will shift slightly and become a band of permitted frequencies under actual pumping conditions.

Experimentally, every parametric amplifier known to the author is pumped at some frequency found in Fig. 21 or close enough to lie within the predictable band width.

If $g_p \to 0$, corresponding to zero perturbation of the original g, then operation is along the $\eta$ axis and there can be no self-excitation, that is, no absorption of energy by the system from the pump. As the magnitude of the periodic perturbation increases ($g_p$ increases) the system operates up in the map where there are broad regions where energy is absorbed from the pump (at $\omega_p$) and drives the system at the frequency $\omega_1$. This, however, is an over-simplification in that $\omega_1$ is not a single frequency.

Since $\omega_1^2 = \frac{g_1}{l_1} = \frac{g_o + g_p \cos \omega_p t}{l_1}$

$$\omega_1 = \sqrt{\frac{g_o}{l_1} + \frac{g_p}{l_1} \cos \omega_p t} = \sqrt{\omega_o^2 + \frac{g_p}{l_1} \cos \omega_p t}$$

$$\omega_1 = \omega_o \sqrt{1 + \frac{g_p}{g_o} \cos \omega_p t} \approx \omega_o \left[ 1 + \frac{g_p}{2g_o} \cos \omega_p t \right] \text{ for } g_p \ll g_o$$

$$\omega_1 \approx \omega_o + \frac{\omega_x^2}{2\omega_o} \cos \omega_p t \text{ where } \omega_x^2 = \frac{g_p}{l_1} \qquad \text{(eq. 15)}$$

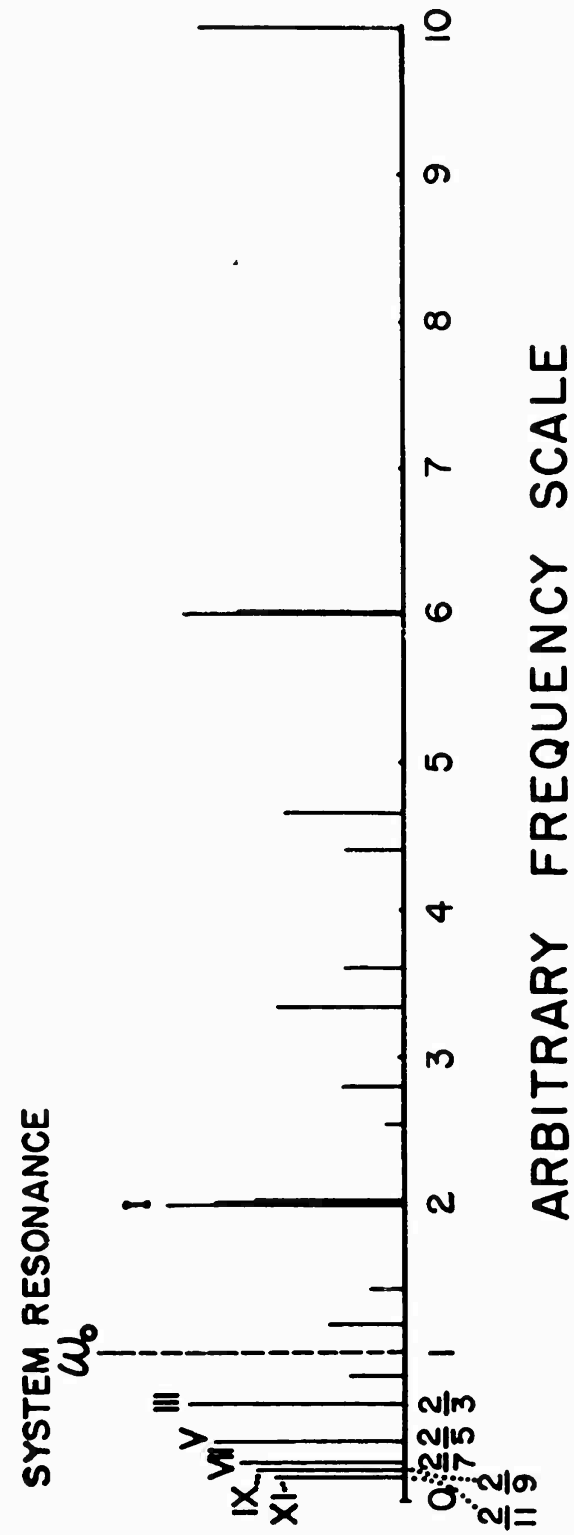


Fig. 21 Permitted Pumping Frequencies

We see the instantaneous radian frequency $\omega_1$ varying about the unmodulated natural frequency $\omega_o$ at the rate $\omega_p$ and with a maximum deviation $\frac{\omega_x^2}{2\omega_o}$ as shown in Fig. 22.

This is frequency modulation. The phase variation for this case is[23]

$$\theta\ (t) = \int \omega_1\ dt = \omega_o t + \frac{\omega_x^2}{2\omega_o\omega_p} \sin \omega_p t + \theta_o \qquad \text{(eq. 16)}$$

$\theta_o$ may be taken as zero by referring to an appropriate phase reference so that the frequency modulated natural frequency is

$$f_o\ (t) = \cos\ (\omega_o t + \beta \sin \omega_p t) \text{ with } \beta = \frac{\omega_x^2}{2\omega_o\omega_p} \qquad \text{(eq. 17)}$$

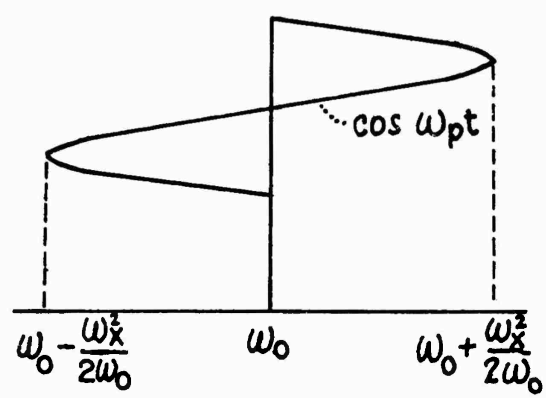


FIG. 22 INSTANTANEOUS FREQUENCY $\omega_1$

$\beta$ is called the modulation index and is the ratio of the frequency deviation to modulating frequency. $\beta$ represents the maximum phase shift of the natural frequency $\omega_o$. When $\beta$ is small as in our case, narrow band frequency modulation results.

$$f_o\ (t) = \cos\ (\omega_o t + \beta \sin \omega_p t)$$

$$= \cos \omega_o t \cos\ (\beta \sin \omega_p t) - \sin \omega_o t \sin\ (\beta \sin \omega_p t)$$

For $\beta << \frac{\pi}{2}$ (narrow band case)

$$\cos\ (\beta \sin \omega_m t) \approx 1$$

$$\sin\ (\beta \sin \omega_m t) \approx \beta \sin \omega_m t$$

Thus

$$f_o(t) \approx \cos \omega_o t - \beta \sin \omega_p t \sin \omega_o t \qquad \beta << \frac{\pi}{2}$$

$$f_o(t) = \cos \omega_o t + \frac{\beta}{2} \cos (\omega_o + \omega_p)t - \frac{\beta}{2} \cos (\omega_o - \omega_p)t \qquad \text{(eq. 18)}$$

This is the expression for amplitude modulation and indicates that the system is being driven at the natural frequency $\omega_o$ and at two sidebands $\omega_o + \omega_p$ and $\omega_o - \omega_p$. In the first instability region pendulum 1 can be pumped at $\omega_p = 2\ \omega_o(1)$ and the system oscillates at three frequencies, $\omega_o$ (1) (its natural frequency) and at two sidebands $\omega_p + \omega_o(1)$ and $\omega_p - \omega_o(1)$. These two conditions produce a spectrum

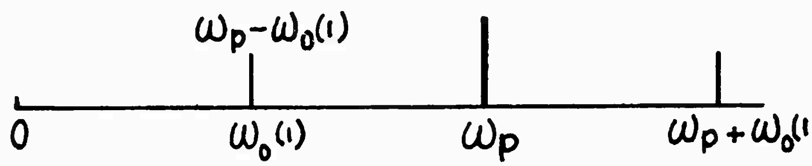


FIG. 23 PUMPING IN FIRST INSTABILITY REGION AT $\gamma = 0$

where $\omega_p - \omega_o(1)$ is identical with $\omega_o(1)$ as in Fig. 23.

Coupling

Now assume a second pendulum as in Fig. 24, pumped by the same electric field variation as the first. Coupling between these two oscillators can be described by normal mode theory[24,25] and a secular equation written. Physically the coupling could be through a number of mechanisms, e.g., torsion of the common support of the two pendula.

The kinetic energies of each pendulum are:

$$T_{11} = m_1 l_1^2$$

$$T_{22} = m_2 l_2^2$$

and there is no kinetic energy of interaction. The potential energy of interaction is assumed to be a function of the difference in their angular displacement $V_{int} = V_{int}(\theta_1 - \theta_2) = 1/2\ k\ (\theta_1 - \theta_2)^2$. The potential energy of the system with no pumping is

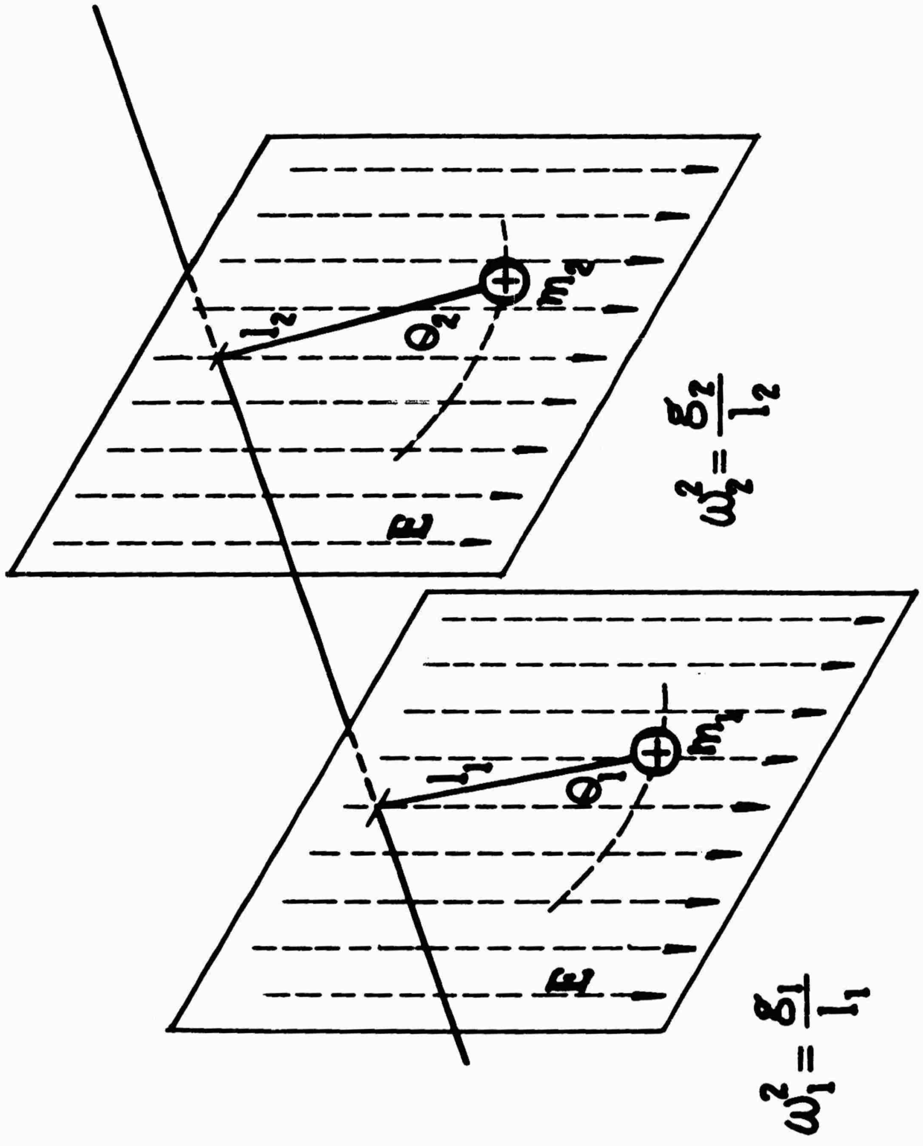


Fig. 24 Two Parametrically Pumped and Position Coupled Pendula

$$V = 1/2\, m_1 g_0 l_1 \theta_1^2 + 1/2\, m_2 g l_2 \theta_2^2 + 1/2\, k\, (\theta_1 - \theta_2)^2 \text{ and} \qquad \text{(eq. 19)}$$

$$V_{11} = m_1 l_1^2 \omega_1^2 + k$$

$$V_{22} = m_2 l_2^2 \omega_2^2 + k \text{ and}$$

$$V_{12} = V_{21} = -k$$

The secular equation is a quadratic in $\omega$

$$\begin{vmatrix} \omega_1^2 - \omega^2 + \dfrac{k}{m_1 l_1^2} & - \dfrac{k}{m_1 l_1^2} \\ - \dfrac{k}{m_2 l_2^2} & \omega_2^2 - \omega^2 + \dfrac{k}{m_2 l_2^2} \end{vmatrix} = 0$$

and the two roots are the two normal frequencies of the system.

$$\omega^2 = 1/2 \left( \omega_1^2 + \omega_2^2 + \frac{k}{m_1 l_1^2} + \frac{k}{m_2 l_2^2} \right) \qquad \text{(eq. 20)}$$

$$\pm 1/2 \sqrt{ (\omega_1^2 - \omega_2)^2 + 2\,(\omega_1^2 - \omega_2^2) \left[ \frac{k}{m_1 l_1^2} - \frac{k}{m_2 l_2^2} \right] + \left[ \frac{k}{m_1 l_1^2} + \frac{k}{m_2 l_2^2} \right]^2 }$$

For small coupling (k small) and

$$\omega_1^2 \neq \omega_2^2$$

$$\omega_+^2 \approx \omega_1^2 + \frac{k}{m_1 l_1^2}$$

$$\omega_-^2 \approx \omega_2^2 + \frac{k}{m_2 l_2^2} \qquad \text{(eq. 21)}$$

This is the usual result indicating a shift of the normal mode frequencies due to the coupling.

Now normal modes can be calculated and if initial conditions such as at $t = 0$, $\theta_1 = \theta_0$, $\theta_2 = 0$ and $\dot\theta_1 = \dot\theta_2 = 0$ are assumed the general solutions to the equations of motion are found to be for the two pendula

$$\theta_1 = \frac{\theta_0}{m_1 + m_2} \left[ m_2 \cos \omega_+ t + m_1 \cos \omega_- t \right]$$

$$\theta_2 = \frac{\theta_0\, m_1}{m_1 + m_2} \left[ -\cos \omega_+ t + \cos \omega_- t \right] \qquad \text{(eq. 22)}$$

For the special case of resonance, that is $m_1 = m_2 = m$ and $l_1 = l_2 = l$

$$\theta_1 \approx \theta_0 \cos \left[ \frac{kt}{2\, m l^2\, \omega_0} \right] \cos \omega_0 t$$

$$\theta_2 \approx \theta_0 \sin\left[\frac{kt}{2\,ml^2\,\omega_0}\right]\sin \omega_0 t \qquad \text{(eq. 23)}$$

where $\omega_0 = \dfrac{\omega_+ + \omega_-}{2}$ and $\dfrac{k}{2\,ml^2\omega_0} = \dfrac{\omega_+ - \omega_-}{2}$

The amplitude of oscillation of each is a function of time and the well known "beat" effect is apparent, as seen in Fig. 25 for the case of resonance. This is a representation of mixed frequencies. That is, only two frequencies are present, $\omega_0$ and $\dfrac{k}{2\,ml^2\omega_0}$ corresponding to $\dfrac{\omega_+ \pm \omega_-}{2}$

Energy in the first pendulum is transferred to the second via the coupling mechanism. After some time which depends on the coupling coefficient k and the "average" frequency $\omega_0$ all of the energy is stored in oscillations of the second pendulum. Now the situation is reversed and the energy is transferred back to the first pendulum.

For slightly detuned pendula the energy exchange still takes place but the initially excited pendulum has a minimum amplitude different from zero[26] as seen in Fig. 26. There has been an incomplete transfer of energy. Coupling is reduced as the pendula are detuned. This is the picture where $\omega_+$ and $\omega_-$ are single frequencies, or more properly where the gravity coefficient is simply $g_0$ for the case of no pumping. Now if we assume that both pendula are pumped by the same electric field variation

$$V = 1/2\, m_1 (g_0 + g_p \cos \omega_p t)\, l_1\, \theta_1^2 + 1/2\, m_2 (g_0 + g_p \cos \omega_p t)\, l_2 \theta_2^2$$

$$+ 1/2\, k\, (\theta_1 - \theta_2)^2$$

$$= 1/2\, m_1 g_0 l_1 \theta_1^2 + 1/2\, m_2 g_0 l_2 \theta_2^2 + 1/2\, k\, (\theta_1 - \theta_2)^2$$

$$+ 1/2\, m_1\, g_p \cos \omega_p t\, l_1 \theta_1^2 + 1/2\, m_2 g_p \cos \omega_p t\, l_2 \theta_2^2 \qquad \text{(eq. 24)}$$

$$V_{11} = m_1 (g_0 + g_p \cos \omega_p t)\, l_1 + k$$

$$V_{22} = m_2 (g_0 + g_p \cos \omega_p t)\, l_2 + k$$

However, now it must be recalled that $\omega_1$ and $\omega_2$ are not single frequencies. The coupled normal mode frequencies are thus

$$\omega_+^2 = (\omega_0^{(1)} + \frac{g_p}{2 l_1 \omega_0^{(1)}} \cos \omega_p t)^2 + \frac{k}{m_1 l_1^2}$$

$$\omega_-^2 = (\omega_0^{(2)} + \frac{g_p}{2 l_2 \omega_0^{(2)}} \cos \omega_p t)^2 + \frac{k}{m_2 l_2^2} \qquad \text{(eq. 25)}$$

For the case of negligibly small coupling (k = o) these expressions reduce to equation 15. That is, as before, each pendulum is being driven at $\omega_0$ and two sidebands $\omega_0 + \omega_p$ and $\omega_0 - \omega_p$.

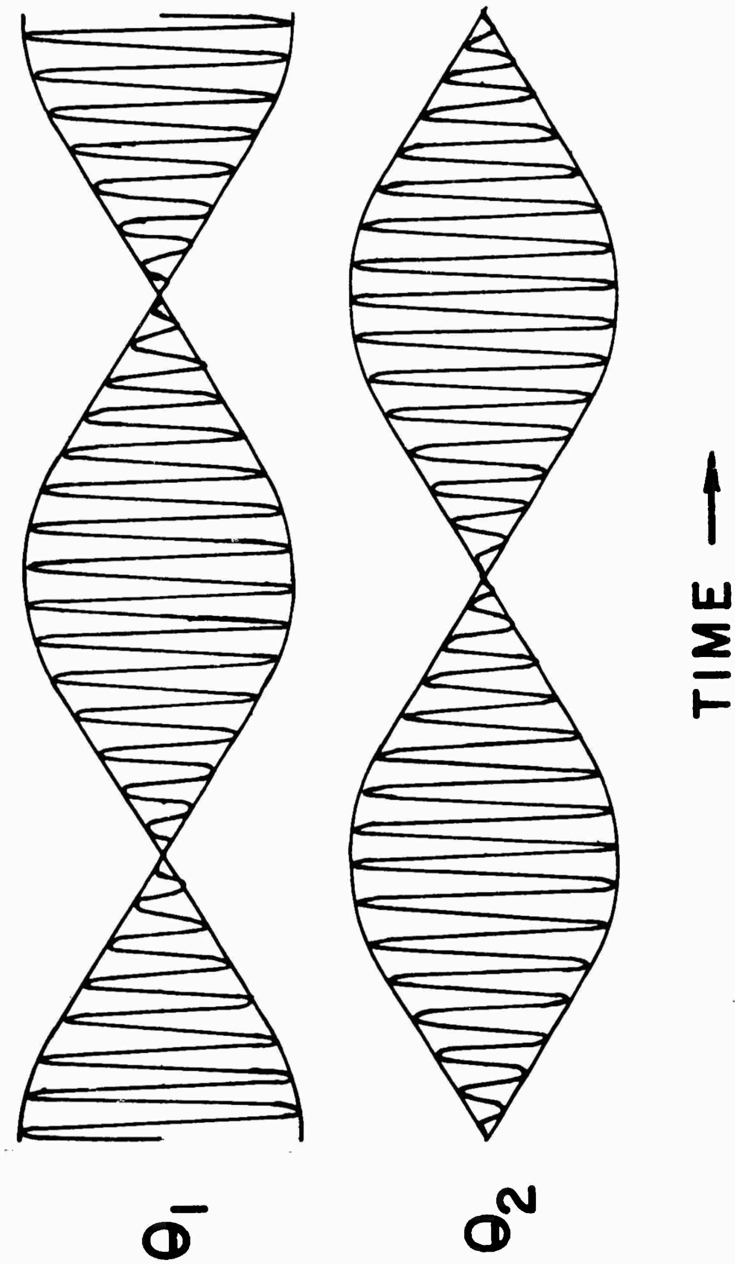


Fig. 25 Plot of Positions of Two Tuned Coupled Pendula

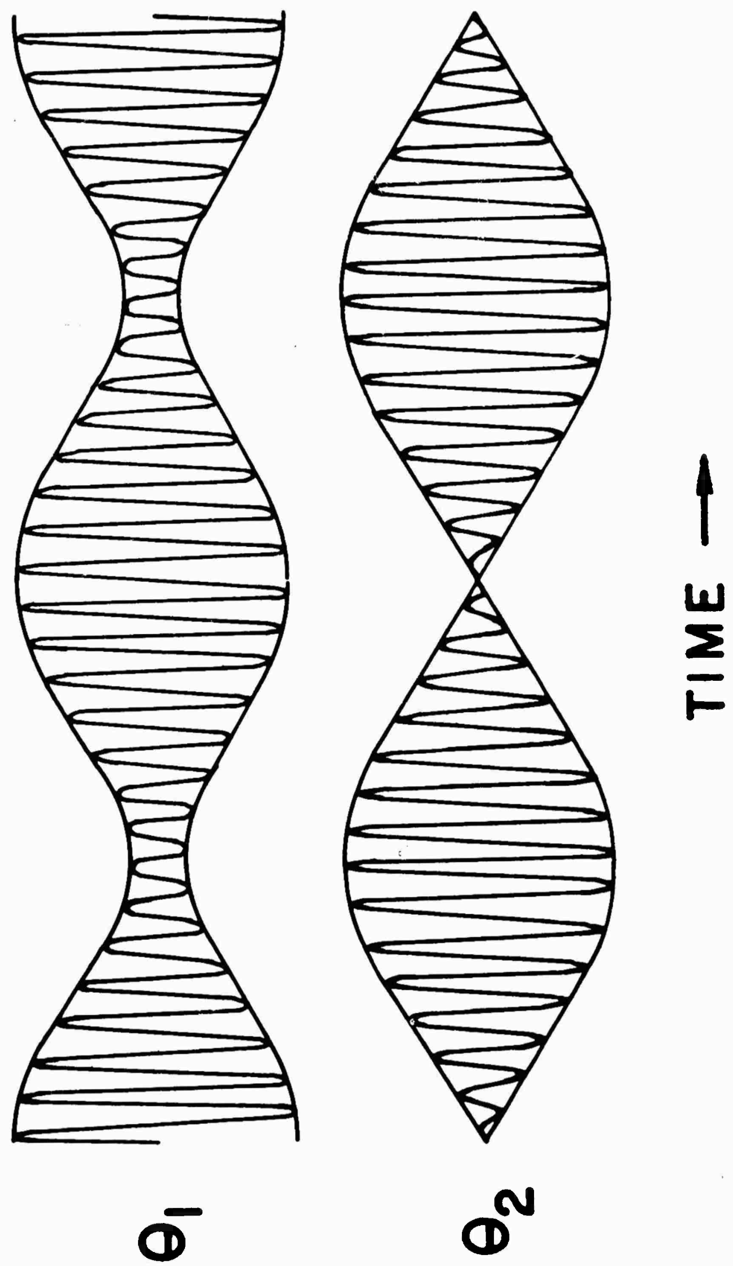


Fig. 26 Plot of Positions of Two Slightly Detuned Coupled Pendula

If the pumping amplitude ($g_p$) is now adjusted, the value of $\nu$ can be varied and operation on the map of Fig. 19 can be moved up to where $\eta$ can be varied slightly from its value of 1 (first instability region) and still remain in the stop band of the system. This permits the natural frequency of either or both pendula to be adjusted slightly from the value $1/2\ \omega_p$. If, for example, $\omega_o(1)$ is adjusted lower than $1/2\ \omega_p$ and $\omega_o(2)$ is adjusted higher by the same amount, the spectrum of Fig. 27 is as shown. Note that $\omega_o(1)$ and $\omega_o(2)$ still add to equal the pump $\omega_p$.

In pendulum 1 energy from the pump source is distributed among the frequencies $\omega_o(1)$, $\omega_p - \omega_o(1)$, and $\omega_p + \omega_o(1)$. Since coupling is very inefficient for detuned pendula the coupling consists principally of transferring power at a certain frequency of pendulum 1 to the same frequency (if permitted) of pendulum 2. The $\omega_p - \omega_o(1)$ mode of pendulum 1 couples to the $\omega_o(2)$ mode of pendulum 2. The $\omega_o(1)$ mode couples to $\omega_p - \omega_o(2)$ mode. The $\omega_p + \omega_o(1)$ mode does not couple with the $\omega_p + \omega_o(2)$ mode. Power is exchangeable, however, between $\omega_o(1)$ and $\omega_p - \omega_o(1)$ as in any amplitude modulation process, the amount of power in the sideband $\omega_p - \omega_o(1)$ being a function of the degree of modulation, that is the magnitude of $\omega_o(1)$ relative to $\omega_p$.

The conditions have now been satisfied for parametric amplification to take place. Consider a certain amount of power which is available at $\omega_o(1)$ frequency as the input power of this amplifier. This power at $\omega_o(1)$ is translated to power at frequencies $\omega_p - \omega_o(1)$ by the pumping process. This is done perfectly efficiently for our lossless case. The power at $\omega_p - \omega_o(1)$ is now coupled to $\omega_o(2)$ and appears as an input power to this pendulum. Again frequency conversion takes place and this energy at $\omega_o(2)$ appears at $\omega_p - \omega_o(2)$ and $\omega_p + \omega_o(2)$. The energy at $\omega_p + \omega_o(2)$ cannot be coupled back to pendulum 1 due to the detuning of $\omega_p + \omega_o(2)$ from $\omega_p + \omega_o(1)$. In circuit theory terms this signal is reactively terminated as in $\omega_p + \omega_o(2)$. The energy at $\omega_p - \omega_o(2)$ can be coupled back to pendulum 1 at frequency $\omega_o(1)$. This feedback effect has been shown from conservation theorems such as those of Manley-Rowe to be positive due to proper phasing for this lower sideband case so that regeneration takes place. If the pumping is strong enough, oscillations occur at $\omega_o(1)$, $\omega_p - \omega_o(1)$, $\omega_o(2)$, and $\omega_p - \omega_o(2)$ without our assumed signal power. This is then an oscillator. If pump power is reduced to where oscillations cease, and the assumed signal power is introduced, that is, a small oscillation is introduced at frequency $\omega_o(1)$ in pendulum 1, there amplification occurs and the amplitude of oscillation is increased. Since either $\omega_o(1)$ or $\omega_o(2)$ can be considered the input, the output can also be taken at either frequency. This is a so-called double channel amplifier which is susceptible to noise input in the unused signal input channel.

For completeness consider the so-called upper-sideband case. Here the natural frequencies of the pendula are adjusted in the instability region of the plot of Mathieu function so that for both pendula they lie on the same side of the $\frac{\omega_p}{2}$ frequency. Now the upper sideband mode of each case couples. However, phase reversal of the signal fed back to the input cause degeneration in this case.

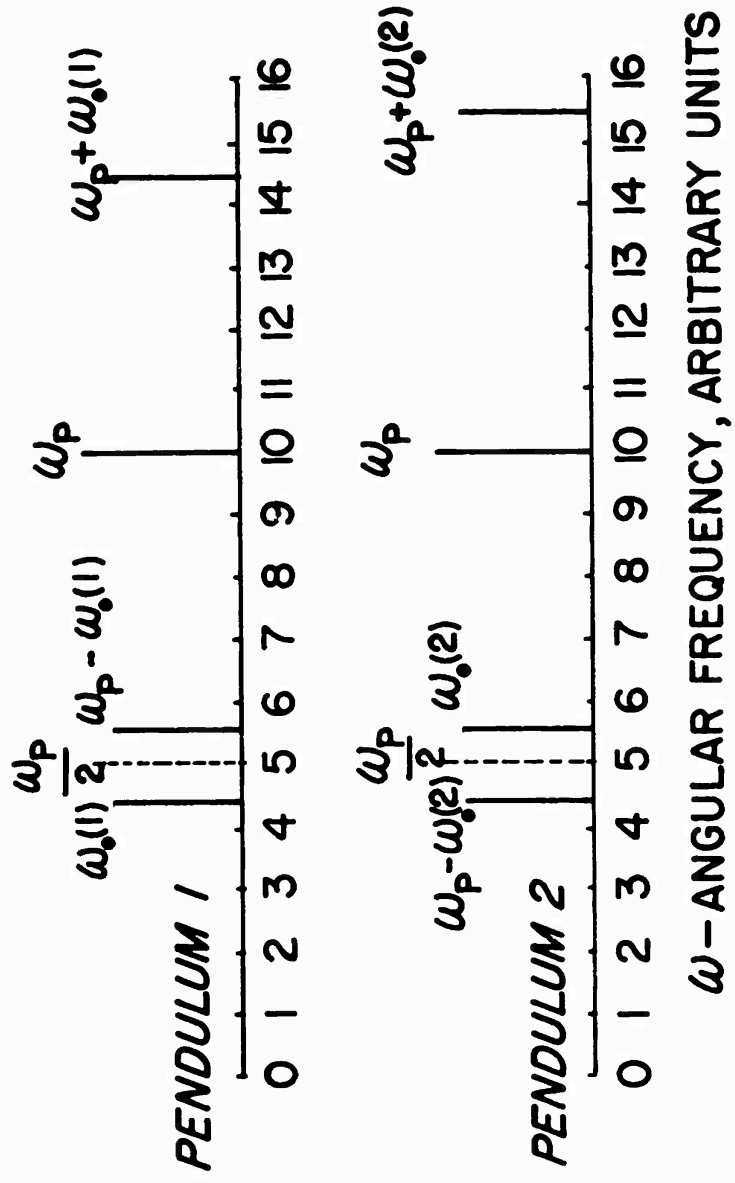


Fig. 27 Spectrum of Coupled Pendula Feedback Process

How far can $\omega_o(1)$ and $\omega_o(2)$ be shifted from the $1/2\ \omega_p$ position? This is determined by the map of instability regions. From Fig. 19 it can be seen that for the extreme $\eta = \gamma$, $\omega_o$ can range from $0.4\ \omega_p$ to $0.67\ \omega_p$. The characteristic exponent $\mu$ falls to zero at these points and reaches a maximum value of approximately -0.25 near $\omega_o = .55\ \omega_p$. This is presented pictorially in Fig. 28. As long as $\omega_o(1)$ and $\omega_o(2)$ both lie within this interval they can be pumped by the same pump frequency $\omega_p$ and couple energy in a manner to cause regeneration. Coupling also falls off rapidly as the two resonant frequencies are separated.

When $\omega_o(1) = \omega_o(2)$ operation is in the degenerate case of parametric amplification and the case of pumping of subharmonic frequencies studied by Rayleigh[27] and others.

The curve of Fig. 19 in general represents the effect of the medium or material on the impringing wave. In the case discussed it represents the effect of the pendulum on the driving wave, i.e., the sinusoidal force field. A $\omega$-$\beta$ diagram of Brillouin diagram can be constructed for this behavior of the wave and will be done to avoid confusion in the succeeding sections when propagating waves are discussed.

Since $\eta$ and $\gamma$ are each inversely proportional to $\omega_p^2$ with a fixed ratio $\frac{\eta}{\gamma} = \frac{g_o}{g_p}$, variation of the perturbing parameter $g_p$ results in a family of straight lines originating at the origin shown in Fig. 19. These lines cut the successive curves at different values of $\beta$. Fig. 29 can then be constructed. As in the usual meaning, the solid lines define frequency pass bands for the wave. The stop bands are the instability regions where attenuation of the wave takes place, i.e., $\mu = \alpha$, and energy is absorbed or given up by the material or in this case, the pendulum. In the pass bands $\mu = i\beta$ and the wave is unattenuated. The width of the stop band and to a lesser extent the position of the stop band is a function of the magnitude of $g_p$. All curves of course can be reduced inside the first zone.

Oscillating Mode Amplifiers

We shall now discuss practical parametric amplifiers that operate on the foregoing principles and constitute an embodiment of the physical mechanisms which have been described.

The Variable-Capacitance Semiconductor Cavity Amplifier, or VARACTOR amplifier is a direct analogue. A voltage sensitive capacitance effect in the depletion layer of a semiconductor when excited with a sinusoidally varying pump voltage becomes a time varying capacitance. Placed in a tuned circuit with an inductance the equation of motion is

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

If the C is varied in time the equation can be written in the form of a

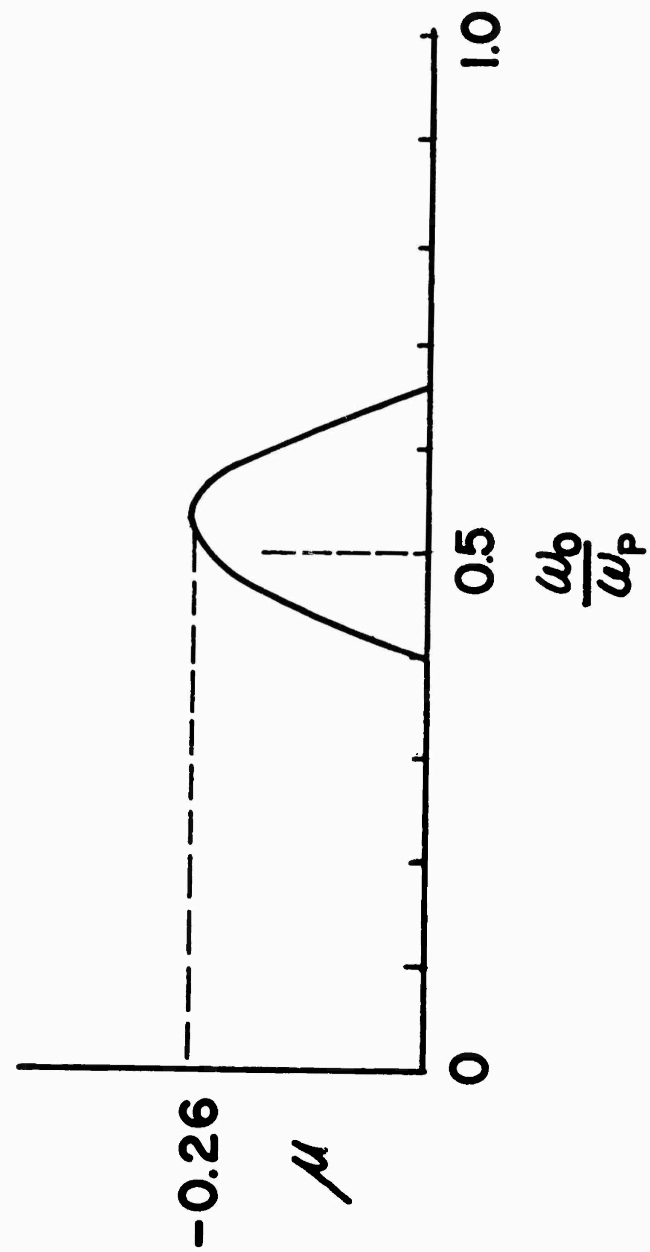


Fig. 28 Band of System Resonant Frequencies which can be Pumped in Instability Region I

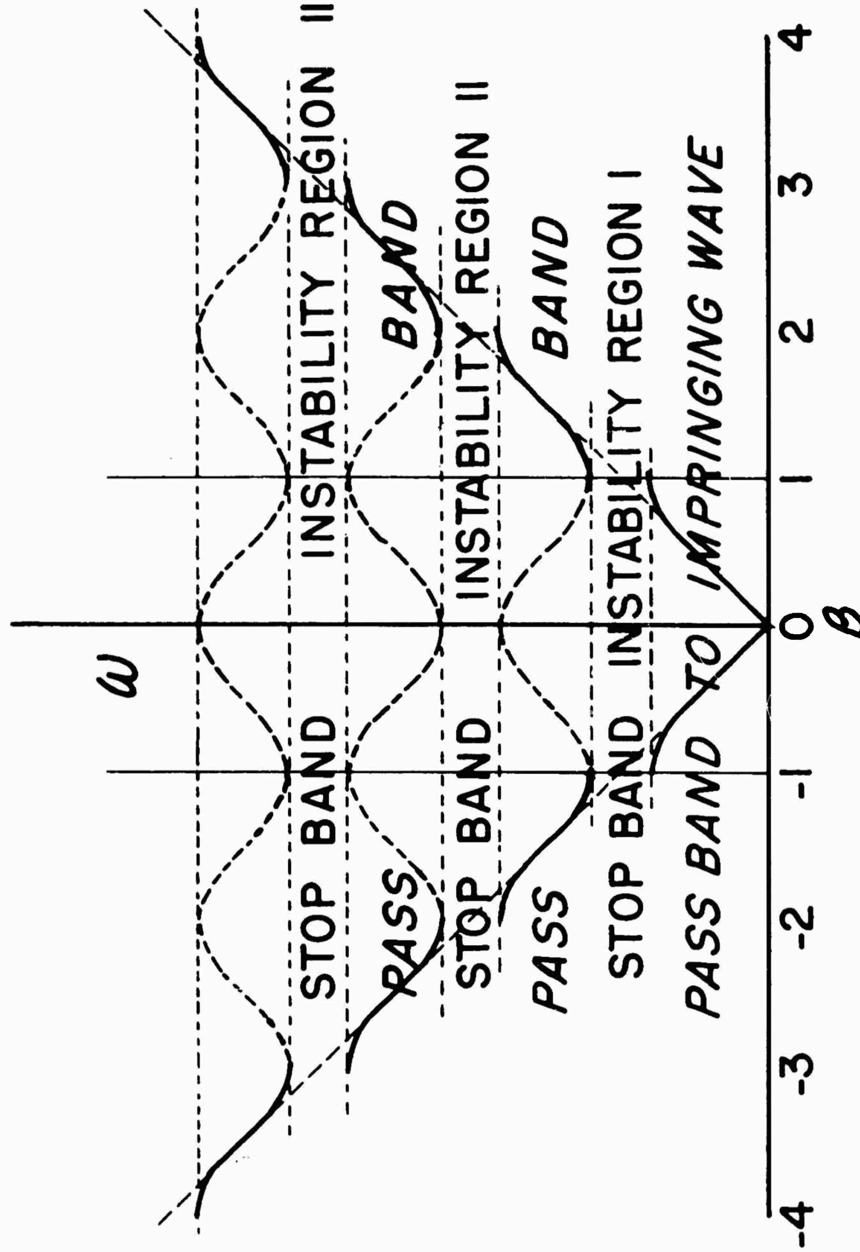


Fig. 29 Brillouin Diagram of Parametrically Pumped Resonant System

**Mathieu's** equation

$$\frac{d^2Q}{dt^2} + \frac{1}{L} \left(A + B \cos \omega_p t\right) Q = 0 \qquad \text{(Eq. 26)}$$

where

$$\frac{1}{C} = A + B \cos \omega_p t$$

If we let $Q = u$ and $\xi = \frac{1}{2} \omega_p t$

**then**

$$\frac{\partial^2 u}{\partial \xi^2} + \left(\eta + \gamma \cos 2 \xi\right) u = 0$$

as before and the map of instability regions, Fig. 19 applies. By direct comparison

$$\frac{g_o}{l} = \frac{A}{L} \quad \text{and} \quad \frac{g_p}{l} = \frac{B}{L}$$

Thus $\omega_o = \frac{A}{L}$ and should be approximately 1/2 of the excitation frequency $\omega_p$, to be pumped in the first instability region.

The evaluation of the constants A and B may be approached by assuming small excitation ($B \ll A$) and a plane condenser[28] whereupon it can be shown that $A \approx \frac{1}{C_o}$, where $C_o$ is the unperturbed capacitance of the non-linear C. In addition $B \approx \frac{4\pi b}{S}$ where S is the surface area of the plane condenser and b is the amplitude constant in the voltage equation

$$e = e_o + b \cos \omega_p t$$

Another approach is to obtain an expression for the functional dependency of capacity on applied voltage for the PN junction. This is largely determined by the density distribution of impurity atoms in the region of the junction. Two common impurity distributions describing the extremes of conditions are so-called "abrupt" and "linear" graded. The capacity of the abrupt junction is

$$C \approx \frac{1}{\sqrt{\Phi - V}}$$

and for the linearly graded is

$$C \approx \frac{1}{\sqrt[3]{\Phi - V}}$$

C = junction capacity

$\Phi$ = barrier potential (determined by material)

V = external applied voltage

If a sinusoidal pumping voltage $V_p$ is applied to either of these non-linear capacitance functions the capacitance varies with time in a manner expressible by an infinite series in C whose coefficients must be determined.

$$C = \sum_{-\infty}^{\infty} C_n e^{jn\omega_p t}$$

If a second sinusoidal voltage $V_s$ is applied to the diode it can be shown[22] that the $n^{th}$ term of the infinite series is responsible for an average power being delivered to the signal force $V_s$ from the pumping force $V_p$ for the condition $\frac{\omega_p}{\omega_s} = \frac{n-1}{2}$

This expression yields the frequency ratios we have obtained from Fig. 19 along the $\eta$ axis. These similar results are obtained by treating the capacitance as a functional non-linearity compared to our equivalent treatment of the capacitance as a time-varying non-linearity. It is easy to give a plausible argument that the two are equivalent.[28] Expand the charge in a Taylor series in voltage

$$q = \sum_{\infty}^{\infty} a_n (v - v_o)^n$$

where $v_o$ is some reference voltage. The capacitance is defined as $\frac{dq}{dv}$ (or $\frac{q}{v}$)

$$C(V) = \frac{dq}{dv} = \sum_{1}^{\infty} A_n n(v-v_o)^{n-1}$$

If the voltage varies sinusoidally

$$v - v_o = b \cos \omega t$$

$$C(V) \rightarrow C(t) = \sum_{1}^{\infty} A_n n \left(\frac{b}{2}\right)^{n-1} (e^{j\omega t} + e^{-j\omega t})^{u-1}$$

$$= \sum_{-\infty}^{\infty} C_n e^{jn\omega t}$$

where $C_n$'s are combinations of $A_n$'s

Therefore the non-linear capacitance is equivalent to the time varying capacitance.

For small voltage variation

$$C \simeq C_o + C_1 \cos (\omega t + \phi) \qquad \text{(Eq. 27)}$$

where $C_1 \ll C_o$

If the non-linear capacitance or time-varying capacitance amplifier is to be anything other than the degenerate case of parametric amplification the resonant frequency of its tuned circuit must be detuned from $\frac{\omega_p}{2}$ if pumping is taking place in instability region I, whereupon an idler or lower side-band frequency appears. This frequency must support real power by being provided with a resonant circuit at the idler frequency composed of the same non-linear capacitance and a separate inductance. If both are near enough to the $\frac{\omega_p}{2}$ frequency, then both can be pumped, coupling takes place, and regeneration occurs as discussed previously.

The Parametrically Pumped Electron Beam Amplifier - Another non-linear medium that can be used to provide parametric mixing between waves is an electron beam. In Fig. 30 the straight dashed line represents the equilibrium position of the electron beam. An individual electron in the beam has an equilibrium position which moves with the velocity of the electron. If the pump field operates in this moving frame of reference, i.e., the pump field travels with the beam and in addition is made to vary sinusoidally then the beam electron is removed from its equilibrium position and oscillates around that position so that its components of motion in any plane containing the beam axis is simple harmonic motion. This is true for both transverse waves (such as cyclotron waves) and longitudinal waves such as slow and fast space charge waves.

This is then a direct anologue to the movement of the charged pendulum bob in the varying electric field. Pumping is possible in terms of the stability map of Mathieu's functions if the pump field moves with the moving frame of reference. This method of pumping has been proposed and is accomplished by feeding the pumping field into the beam from some parallel external structure. The more normal mode of operation is to utilize the fast space charge wave as a pump whereupon the device becoming a propagating system which is covered in the next section on pumping and coupling in propagating systems.

If a signal is introduced into the beam as a fast space charge wave, waves of sideband frequencies are generated and propagate down the beam as fast space charge waves. Coupling occurs between electrons via their forces of confinement in the beam and mutual repulsion. The signal and lower side-

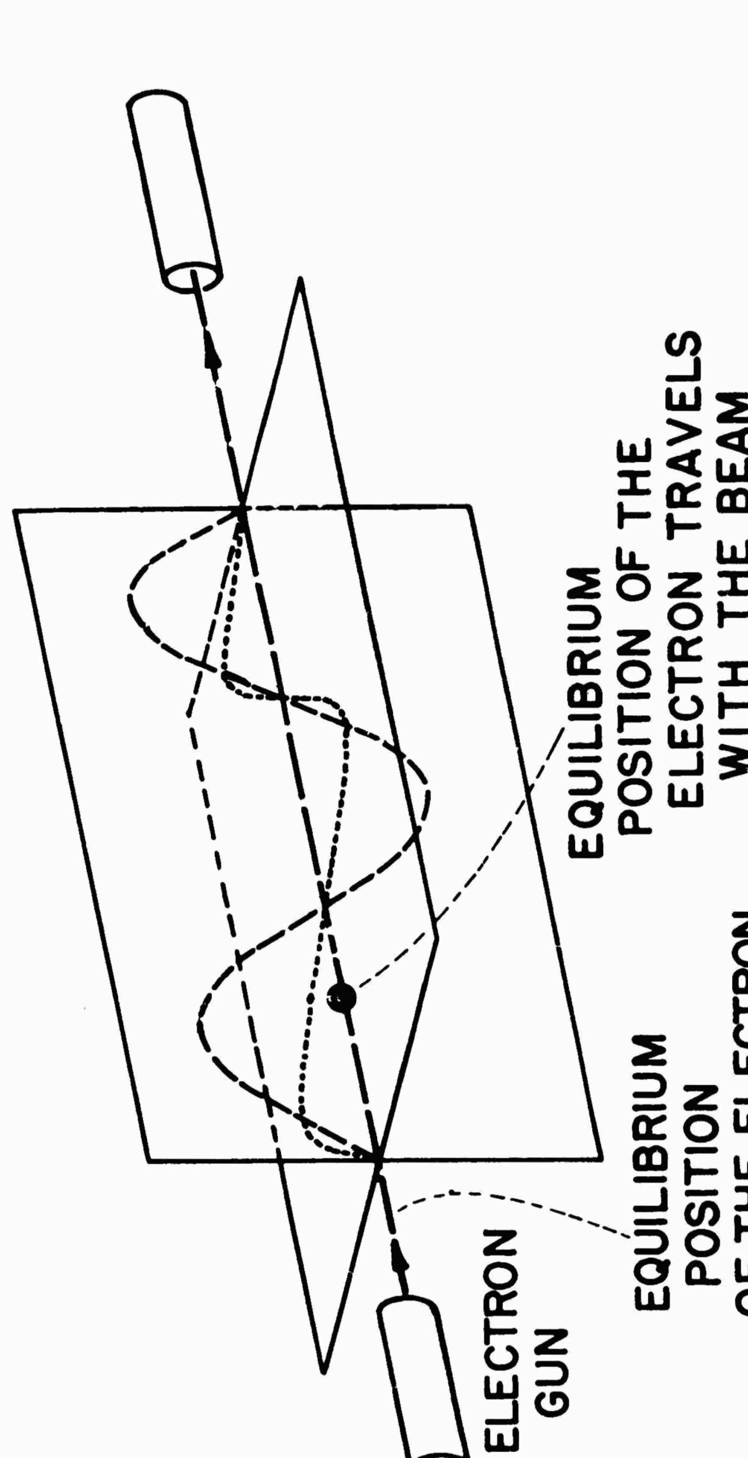


Fig. 30 Schematic of Electron Beam Amplifier

band (idler) wave then grow exponentially with distance down the beam. These will be recognized as the basic parametric principles discussed above. However, the analogue is more complete for a transmission line.

The Ferromagnetic Resonance Amplifier - (Ferrite parametric amplifier or ferromagnetic maser) employs non-linear effects occurring in the precessing magnetization of magnetic resonance phenomena as the non-linear medium to be used to provide parametric mixing between waves.

In the ferromagnetic material where coupling between atomic moments is strong a macroscopic magnetization vector $\vec{M}$ can be defined as the vector sum of the atomic dipole moments per unit volume. The equation of motion of $\vec{M}$,

$$\frac{\partial \vec{M}}{\partial t} = \gamma\,(\vec{M} \times \vec{H}) \qquad \text{(Eq. 28)}$$

where $\gamma = ge/2_{mc}$, e/m is the charge to mass ratio for the electron, c is the velocity of light and g is the Laude' spectroscopic-splitting factor, states that if the material with magnetization $\vec{M}$ is placed in a d.c. magnetic field $\vec{H}$ the torque $\vec{M} \times \vec{H}$ causes $\vec{M}$ to precess about $\vec{H}$ at a constant angular velocity $\omega_0 = \gamma H$.

Dissipative forces are introduced in a phenomenological manner such as the formulation of Landau and Lifshitz

$$\frac{\partial \vec{M}}{\partial t} = \gamma\,(\vec{M} \times \vec{H}) - \frac{\Lambda}{M^2}\,(\vec{M} \times \vec{M} \times \vec{H})$$

where $\Lambda \ll \gamma M$ for small damping. The short range exchange torque for a material with cubic symmetry can be expressed[30] as

$$\vec{M} \times \vec{H}_{ex} = \frac{2A}{M^2}\,(\vec{M} \times \nabla^2\ \vec{M})$$

and added to the equation of motion

For an idealized lattice with only exchange torques present

$$\frac{\partial \vec{M}}{\partial t} = \frac{2\gamma A}{M^2}\,(\vec{M} \times \nabla^2\,\vec{M}) \qquad \text{(Eq. 29)}$$

For small excitation $\vec{M} = \vec{M}_s + \Delta\vec{M}$. Differention with respect to time

$$\frac{d^2(\Delta\vec{M})}{dt^2} = -\left[\frac{2\,\gamma\,A}{M_s}\right]^2 \nabla^4\ (\Delta\vec{M})$$

This is a wave equation with solutions

$$\Delta\vec{M} = \Delta\vec{M}_0\ e^{j(\omega t + \vec{k}\cdot\vec{r})}$$
$$\omega = \frac{2\gamma A}{M_s}\ k^2 \qquad \text{(Eq. 30)}$$

where long range magnetic dipole interaction between spins is neglected.

These excited states of the system are called spin waves. The magnetization is not the same at all points of the sample. If there is a spatial periodicity in the angle between the direction of the magnetization vector and the direction of the polarizing field then a graph of this angle against position looks like a wave profile. It travels through the medium and at any point there is a sinusoidal time dependence.

The above wave equation represents a plane wave. If the spin wave wave length is sufficiently short ($k > k'$ in Fig. 31) the waves can be represented as plane waves in an infinite medium with negligible effect from the sample boundary. In Fig. 31 the two solid curves correspond to spin waves propagating along and at right angles to the direction of magnetization. As $k \to o$, $\lambda \to \infty$ the assumptions leading to equation 30 are violated. Principally the boundary conditions now become important; a plane wave representation is not correct, the exchange forces exert only a slight influence on the motion of the spin; the characteristic frequency is then nearly independent of the exchange field and consequently is nearly independent of k. The frequency is now dependent on the long range dipolar interaction of the spins and the static magnetic field. As wave lengths approach sample size the correct normal modes are the magnetostatic modes investigated by Walker[31]. These "standing waves" depend on the shape of the body but not its size, provided the body is large enough for exchange torque to be neglected yet small enough for propagation effects to be neglected. Magnetostatic modes contain the $k = 0$ spin wave or principle precessional mode of magnetic resonance where the magnetization vector, magnitude and direction are the same throughout the material. The lower frequency limit $\omega_1$ of magnetostatic modes coincides with that of spin waves. The upper frequency limit of magnetostatic modes $\omega_3$ is greater than the upper limit of spin waves, $\omega_2$. Fig. 31 shows an approximate representation of the density of magnetostatic modes for a spherical sample[32].

A degeneracy of the frequency of the mode of uniform precession with spin waves of shorter wave length can be seen from the Fig. 31. Energy can be transferred from the uniform precessional mode to the spin waves and lost to the system. Several types of coupling occur but the one of interest to us is due to the non-linear behaviour of the spin system at pump fields of large magnitude. As pump power increases beyond a critical value, the uniform precessional mode broadens and weakens in intensity as certain spin waves grow at the expense of the uniform mode and generally a subsidiary resonance occurs at a field strength below the main resonance. This coupling to the long wave length spin waves of magnetostatic modes is the process of conversion of power to other frequencies which is the basis of parametric amplification.

Following the principles of parametric amplification laid down in previous sections, a ferromagnetic parametric amplifier is now r ealizable. First, the conservations theorems must be obeyed by permitting real power to exist at signal, idler, and pump frequencies. This can be accomplished in at least three ways. If the microwave cavity is resonant at the signal

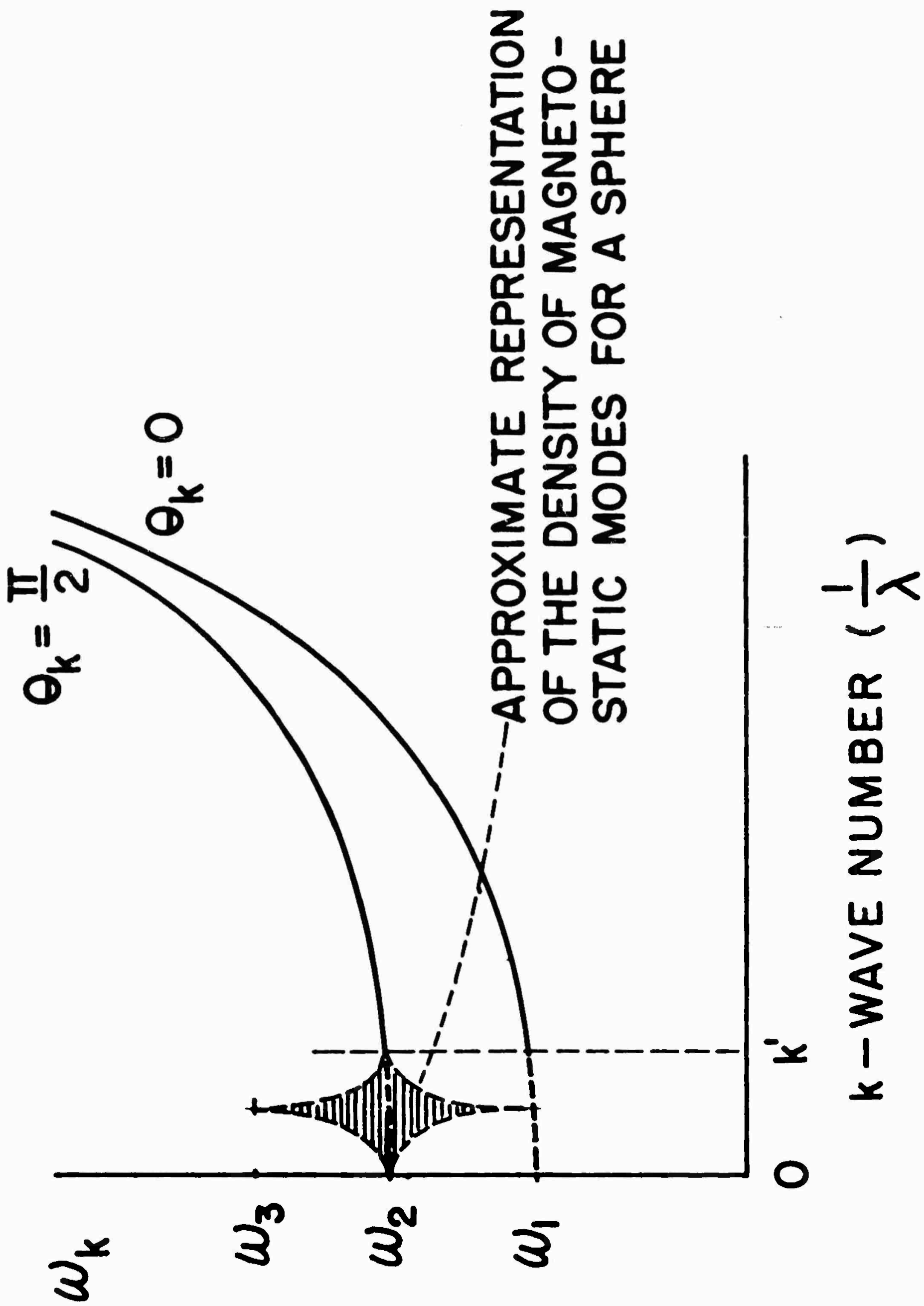


Fig. 31 Frequency of Spin Wave, $\omega_k$, as Function of Wave Number k

and idler frequencies and the pump frequency corresponds to the principle precessional mode or another magnetostatic mode then operation is obtained in the "electromagnetic" mode of Suhl[33]. If two modes at magnetostatic and one is a cavity mode, this is the "semistatic" mode of Suhl. If all three modes are magnetostatic modes then it is called "magnetostatic." In all cases the signal and idler frequencies must add up to the pump frequency by selecting the proper magnetostatic modes for the 3 frequency regenerative amplifier-pumped in the first instability region of the Mathieu function. Pumping in other regions is also possible. Coupling between the two resonance systems takes place via the mechanism discussed under the feedback section and is identifiable as certain cross product terms in the expanded equation of motion for two driving frequencies, the signal and pump frequency.

## PHYSICAL MECHANISMS OF PUMPING AND COUPLING IN PROPAGATING SYSTEMS

To avoid the narrow-bandedness of the resonant systems of the preceeding section, a propagating system can be employed. Conceptually this could be a cascaded sequence of resonant systems with minimum gain and maximum band width per section. Stagger-tuning could also be employed. This approach of treating coupled cavities has been analyzed[34] and utilized[35]. As the number of coupled elements is increased and the effect on the transmission line per element is decreased the periodically loaded line is obtained and in the limit a continuous line is approached. Compatible results should then be obtained between the coupled cavity approach the periodically loaded line and the continuous line approach. The periodically loaded coupled transmission line has been analyzed by Tien and Suhl[36] and Tien[37].

In the propagating system parametric amplifier a transmission line propagates two forward normal modes at frequencies $\omega_1$ and $\omega_2$. Some parameter of the line such as the shunt capacitance or series inductance is time-varying due to a pump field at $\omega_p$. If the parameter is varied properly a pump wave will propagate down the line causing the parameter to be a function of time and distance. The pump wave will couple the two frequencies $\omega_1$ and $\omega_2$, now seen to be the signal and idler frequencies $\omega_s$ and $\omega_-$, and cause them both to grow exponentially. Reverse normal modes are also possible.

### The Conservation Theorems

The power frequency formulas discussed previously must be modified to deal with distributed systems. Only the real power and non-linear reactance formulas, or Manley-Rowe equations will be considered.

In distributed systems the energy is stored or dissipated over space while power enters and leaves at the boundary, i.e., through the ports. The quantities of interest are the powers entering and leaving the system at each frequency.

Assuming each power is a divergence

$$P_a = - \nabla \cdot \vec{F}_a \qquad \text{(Eq. 31)}$$

then the power frequency formula

$$\sum_a \frac{P_a}{\omega_a} = 0$$

can be written using the divergence theorem in the form[8],

$$\sum_a \frac{\int_s \vec{F}_a \cdot \vec{n}\, dS}{\omega_a} = 0 \qquad \text{(Eq. 32)}$$

where the integration is over the surface and $\vec{n}$ is the unit vector pointing inward perpendicular to the surface. $\vec{F}_a$ is interpreted as the contribution to time-average value of the power-flow vector $\vec{F}$ caused by frequency $\omega_a$.

Expressing P as a divergence has been done for the non-linear, dispersion-free, anisotropic, inhomogeneous, reciprocal, stationary, electromagnetic media. The resulting proof is that $\vec{F}_a$ can be written

$$F_a = \frac{1}{2} R_e \vec{E}_a \times \vec{H}_a^* \qquad \text{(Eq. 33)}$$

which is average power in complex notation. The Manley-Rowe formulas can then be written in the form of equation 32, where $\vec{F}_a$ is the Poynting vector or power flow vector, $\bar{E} \times \bar{H}$ at the frequency $\omega_a$ and the integral

$$\int_s (\vec{E} \times \vec{H}) \cdot \vec{n} ds \qquad \text{(Eq. 34)}$$

is the energy flow through a surface in a distributed system per unit time; the integration being taken over the surface.

As before the conservation theorems are then expressions of the conservation of energy.

Duals in Resonant System Case

In the oscillator certain events took place after a certain time. In the propagating system events occur after a certain distance so that distance plays the role of time. Frequency was all important in the oscillating system. Now wave length is the fundamental quantity of interest in the distributed, guided system. As is customary, the inverse of wave length (multiplied by $2\pi$) the propagation constant, $\beta$, is employed and the relation between wave length and frequency, being a function of the structure, is specified by some means such as the $\omega$ - $\beta$ or Brillouin diagram of Fig. 32.

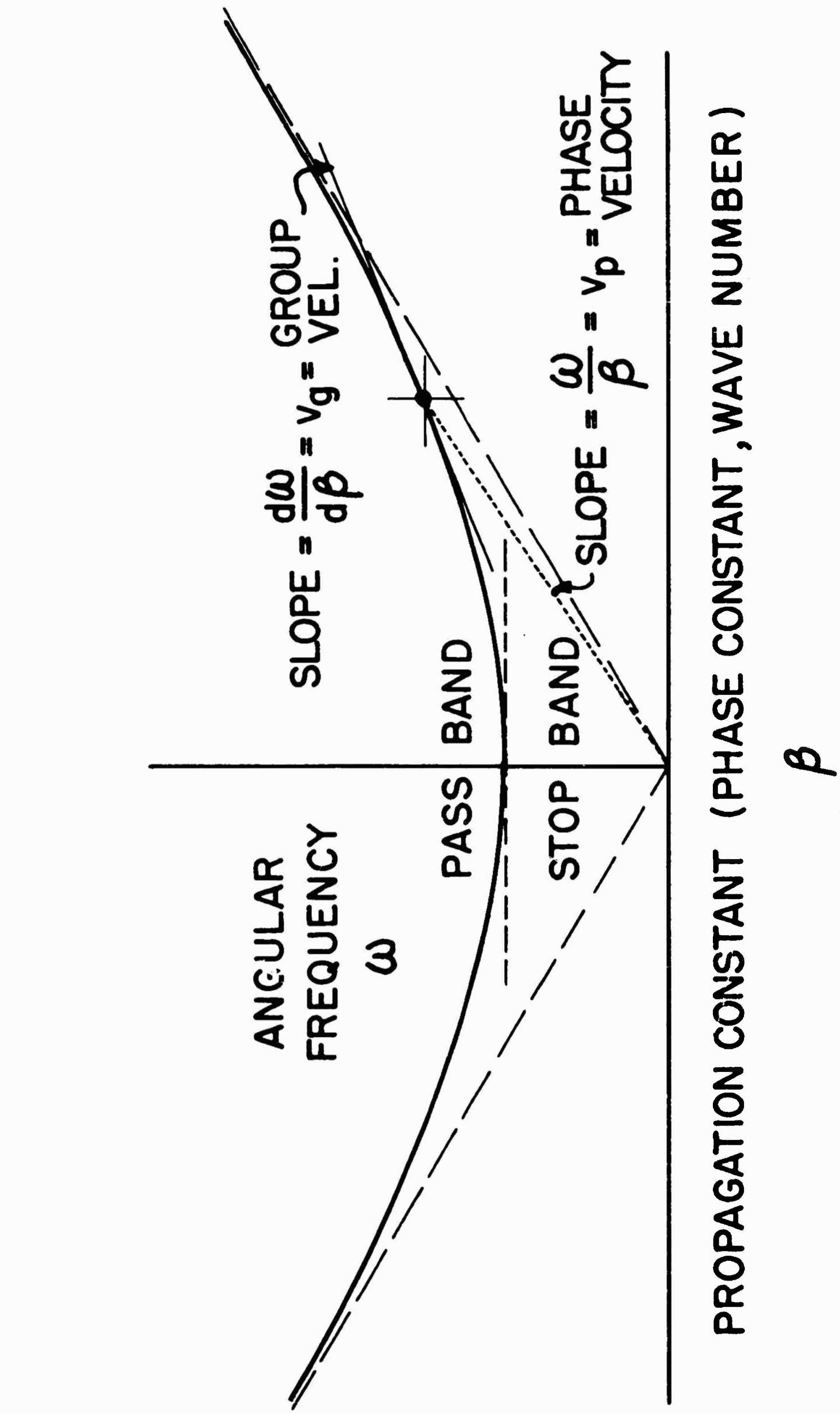


Fig. 32 Brillouin Diagram for Propagating System

The slope to a point on the characteristic is the phase velocity, while the incremental slope of the characteristic is the group velocity as shown.

It is desired that wanted frequencies fall in the pass band of the structure while unwanted frequencies fall in the stop band.

Pumping

The laws relating the pump and system resonant frequencies are the same as for the resonant system case when terms are properly defined.

The traveling wave systems are generally pumped in the first instability region of the Mathieu's function, i.e., where $\omega_p = \omega_s + \omega_i$ and normally the system is in the degenerate case, $\omega_p = 2\omega_o$. Only Fredricks[38] obtains, theoretically, pumping frequencies corresponding to the other instability regions for the traveling wave case but concludes that they are inefficient compared to the above case. The signal and idler frequencies fall in pass bands, while the upper sideband falls in a stop band.

Coupling

As coupling occurred in the resonant frequency case only for systems nearly tuned to each other, i.e., which had the same frequency, $\sec^{-1}$, here coupling occurs following the dual mentioned above, only when two waves have nearly the same $(\text{distance})^{-1}$ or inverse wave length or wave number, $\beta$. Consequently, the condition $\beta_p = \beta_s + \beta_i$ is also necessarily fulfilled for forward traveling wave amplification while $\beta_p = -\beta_s + \beta_i$ is required for backward traveling wave amplification. For the condition $\beta_p = \beta_s + \beta_i$ to hold for a band of frequencies it is also necessary that the incremental slope of the characteristic curve of the signal wave be the same as that of the idler wave.

$$\left[\frac{d\omega}{d\beta}\right]_i = \left[\frac{d\omega}{d\beta}\right]_s \qquad \text{(Eq. 35)}$$

The traveling wave interaction mechanism can now be shown graphically. (see Currie and Gould[39])

The non-linear parameter of the structure is varied by a pump field at $\omega_p$ frequency. The phase shift per section of the structure is $\beta_p$ and can be adjusted along with the amplitude of the pump by adjusting the pump. The phase shift of the signal wave $\beta_s$ at frequency $\omega_s$ is determined by the shape of the Brillouin diagram. If the pass band of the structure is centered at $\frac{\omega_p}{2}$ and $\beta_p = \pi$ the situation illustrated in Fig. 33 exists.

The solid curve is the signal characteristic while the dashed curve is the idler characteristic. They coincide. Gain is bilateral and exists for every frequency in the pass band since $\beta_p = \beta_s + \beta_i$ and $\left(\frac{d\omega}{d\beta}\right)_s = \left(\frac{d\omega}{d\beta}\right)_i$ everywhere in the pass band. With bilateral gain the amplifier is

potentially unstable if reflected waves are present. It is operable for only one specific phase relation as was the degenerate case for the resonant systems.

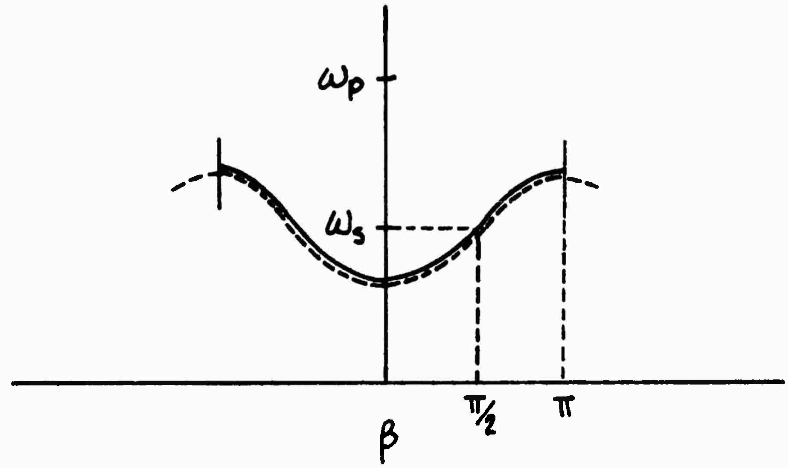


FIG. 33 DEGENERATE CASE IN PROPAGATING SYSTEM

If the pump frequency is reduced slightly below the $2\omega_s$ with the same phase shift, the idler curve is reduced in frequency as shown in Fig. 34.

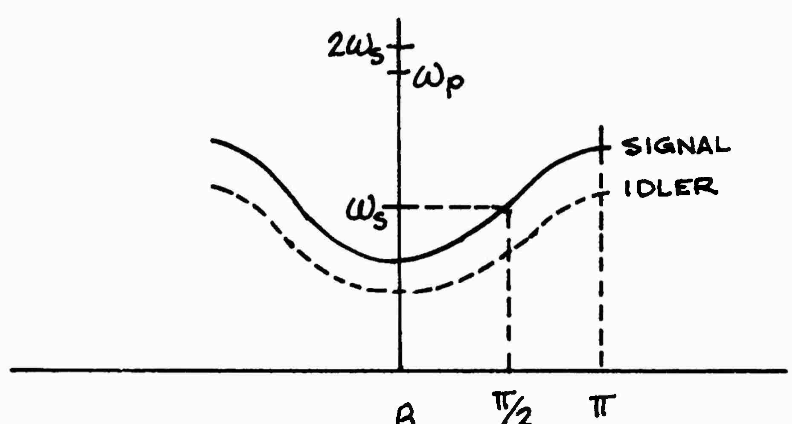


FIG. 34 NON-DEGENERATE CASE IN PROPAGATING SYSTEM WITH IMPROPER PHASE RELATION

The amplifier is still bi-lateral but the gain is reduced since the maximum gain conditions are not satisfied. The band width over which amplification can occur is reduced. This band width is measured by the vertical overlap of the signal and idler curves.

Finally, following Currie and Gould, assume that the pump frequency is less than $2\omega_s$ but that the phase shift per section $\beta_p < \pi$. Recalling from the discussion of modulation that the carrier or pump phase shift is added to the side bands, the new idler curve is not only shifted downward but is now displaced horizontally to the left due to the change in pump phase shift as shown in Fig. 35. The gain is now different in the forward and backward directions. The conditions $\beta_p = \beta_s + \beta_i$ and $\left(\frac{d\omega}{d\beta}\right)_s = \left(\frac{d\omega}{d\beta}\right)_i$ are fulfilled in the interaction region denoted by the circle in Fig. 35. Also shown is the upper side band curve which is presumed to fall on a stop band of the structure to fulfill the Manley-Rowe equations for a three frequency lower side band parametric amplifier.

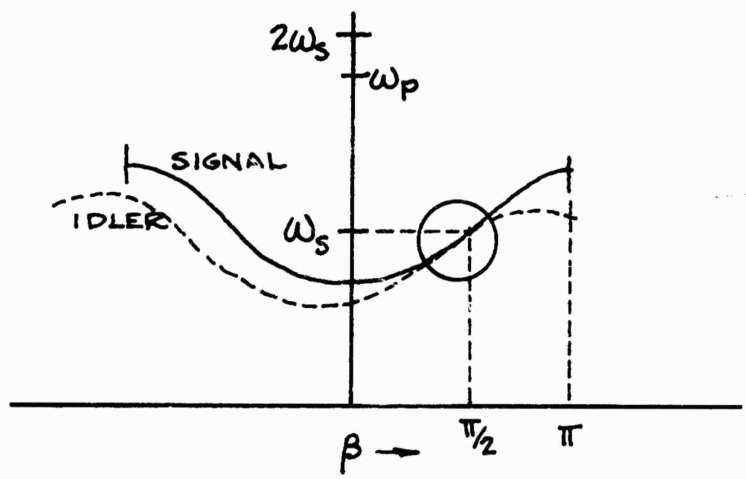


FIG. 35 NON-DEGENERATE CASE IN PROPAGATING SYSTEM WITH PROPER PHASE RELATION

The signal and idler waves in the forward direction are strongly coupled to form a conjugate pair of waves at each frequency that are growing and decaying. These are the two normal modes of the system in the forward direction. Two other normal modes exist for the backward direction but are uncoupled.

This picture of the interaction mechanism must be modified in several ways. Grabowski and Weglein[40] have shown that the shape of the Brillouin diagram is altered when waves couple and may strongly influence the usable bandwidth and other properties. The Brillouin diagram of Fig. 29 represents pass and stop bands presented to the pumping source by the active element. The region of interest here is the stop band where energy is absorbed by the pump. In a propagating system a Brillouin diagram is obtained for the propagating structure. Here the region of interest is the pass band which must pass the desired frequencies (the signal and the idler or lower sideband frequencies and possibly the pump frequency).

Somehow the pump frequency and phase must be introduced to efficiently mix with the signal. Two possibilities suggest themselves. One, the structure could have a pass band at the pump frequency. The pump energy would then propagate down the system. In effect then the stop band of the active element must coincide with a pass band of the structure and the energy absorption of a section of the active element must be sufficiently small to permit pump energy to pass on to succeeding sections. A second possibility is to introduce a pump voltage of equal amplitude and proper phase to each section of the active element. This is a parallel pumping scheme. No structure pass band at the pump frequency is then needed.

It has been indicated that the mechanism of pumping and coupling in propagating systems can be understood in terms of the theory for resonant systems if the duals of the two systems are used. Many of the advances in each field have taken place without cross reference to the other field. Consequently any unifying treatment is desirable and profitable.

The discussion has been limited in the main to the three-frequency parametric phenomena. Extention to the four (and more) frequency phenomena is in order but cannot be attempted in the space allowed.

## SUMMARY

A working definition of an amplifier has been employed with an "agent" driven by a local source of power. When the local power is alternating, a condition imposed by the nature of certain agents, the amplifying process is involved in, but not synonymous with the modulation process. All amplifiers of this type are considered to obey common principles which justifies their classification together under the generic term of parametric amplifier.

The power-frequency formulas or conservation theorems derived from the conservation of energy govern the allowed power at a given frequency in the modulation process. Only systems involving three frequencies are discussed.

A feedback mechanism which dictates unlimited gain in the lower sideband case and limited gain in the upper sideband case is described. Feedback is found to be positive in both cases but with a loop gain greater than one in the lower sideband case and less than one in the upper sideband case.

The pumping of a resonant system parametrically is found to follow the stable and unstable solutions of a Mathieu differential equation describing the motion of the system. An entire spectrum of pumping frequencies is obtained both above and below the system resonant frequency. The spectrum consists of discrete bands of permitted pumping frequencies some of which have been verified experimentally by workers in the field.

Coupling between two resonant parametrically pumped systems gives rise to sidebands which fulfill the requirement for positive feedback effects to occur causing the limited or unlimited gain conditions.

Three oscillating mode amplifiers are discussed as embodiments of these principles: The Variable-Capacitance Semiconductor Cavity Amplifier, The Parametrically Pumped Electron Beam Amplifier, and the Ferromagnetic Resonance Amplifier.

The physical mechanisms of pumping and coupling in systems with propagating waves are described in terms of duals of the resonant system case. The requirements on frequency relations are the same as before, while the new requirements on phase and distance are shown graphically.

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| Bell Telephone Laboratories<br>Allentown, Pennsylvania<br>Attn: J. J. Suozzi | |

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Chief, Bureau of Naval Weapons
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SUBJECT ANALYSIS OF REPORT

| DESCRIPTORS | CODES | DESCRIPTORS | CODES | DESCRIPTORS | CODES |
|---|---|---|---|---|---|
| Microwave | MICR | Mechanisms | MECH | | |
| Parametric | PARA | Frequencies | FREQ | | |
| Amplification | AMPF | Amplifiers | AMPL | | |
| Theory | THEY | Electronics | ELCO | | |
| Conservation | COSE | Sidebands | SIDB | | |
| Feedbacks | FEED | Modulation | MODU | | |
| Pumping | PUMP | | | | |
| Resonant | RESN | | | | |
| Circuits | CIRC | | | | |
| Coupling | COUP | | | | |
| Propagating | PROO | | | | |
| Systems | SYST | | | | |

PRNC-NOL-5070/28 (5-62)

Naval Ordnance Laboratory, White Oak, Md.
(NOL technical report 62-102)
THE NATURE OF MICROWAVE PARAMETRIC AMPLIFICATION (U), by Edward T. Hooper, Jr. 5 Sept. 1962. 71p. charts, tables. BuWeps task RMWC-51-025/212-1/ F009-04-001. UNCLASSIFIED

The principles of parametric amplification are discussed. Included are the conservation theorems that place restraints on powers at various frequencies, feedbacks operating internally, the physical mechanisms of pumping resonant circuits, and the physical mechanisms of pumping and coupling in propagating systems.

Abstract card is unclassified

1. Amplifiers, Parametric
2. Amplifiers, Microwave

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II. Hooper, Edward T. Jr
III. Project

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